ON THE INTEGRALITY OF AN EXTREME SOLUTION TO PLUPERFECT GRAPH AND BALANCED SYSTEMS

R. CHANDRASEKARAN

University of Texas at Dallas, Box 688, Richardson, TX 75080, USA

A. TAMIR

Department of Statistics, Tel Aviv University, Tel Aviv, Israel

Received August 1983
Revised June 1984

Let A be a nonnegative integer matrix, and let e denote the vector all of whose components are equal to 1. The pluperfect graph theorem states that if for all integer vectors b the optimal objective value of the linear program $\min\{e'x \mid Ax \geqslant b, x \geqslant 0\}$ is integer, then those linear programs possess optimal integer solutions. We strengthen this theorem and show that any lexicomaximal optimal solution to the above linear program (under any arbitrary ordering of the variables) is integral and an extreme point of $\{x \mid Ax \geqslant b, x \geqslant 0\}$. We note that this extremality property of integer solutions is also shared by covering as well as packing problems defined by a balanced matrix A.

perfect graphs * balanced matrices * integral extreme points

The pair (A, c) of an $(m \times n)$ integral matrix A and an integral n-vector c is said to be totally dual integral (TDI) if

$$\min\{c'x \mid Ax \geqslant b, x \geqslant 0\} = \min\{c'x \mid Ax \geqslant b, x \geqslant 0, x \text{ integer}\} \qquad (1)$$

for all integral b for which the linear program has optimal solutions.

Clearly, if (A, c) is TDI, then the optimal value of the objective of the linear program is integral for all integral b for which it exists. The converse statement is not always true.

Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad c = (2, 2, 2),$$

then the optimal objective value of $\min\{c'x \mid Ax \geqslant b, x \geqslant 0\}$ is integral for all integer b. However, (A, c) is not TDI. (Try $b = (1, 1, 1)$.)

If we restrict our attention to nonnegative matrices A and set $c = (1, \dots, 1)$ then the above converse statement is true and is known as the pluperfect graph theorem [1,2].

Theorem 1. *Let A be an $m \times n$ nonnegative integral matrix with no zero rows. For any integer b define*

$$f(b) = \min\left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0 \right\}.$$

Suppose that $f(b)$ is integer for all integer vectors b. Then for any integer vector b there exists an integer vector $x^0 = x^0(b)$ such that

$$Ax^0 \geqslant b, \quad x^0 \geqslant 0 \quad \text{and} \quad f(b) = \sum_{j=1}^{n} x_j^0.$$

In this work we strengthen the above theorem and show that any lexicomaximal optimal solution to the linear program (under any arbitrary ordering of the variables) is integral and is an extreme point of $\{x \mid Ax \geqslant b, x \geqslant 0\}$.

We discuss certain algorithmic implications of this result and show how to find such an optimal integer extreme point.

Finally we note that this extremality property of integer solutions holds also for covering as well as packing problems defined by a balanced matrix A.

To facilitate the discussion let Π denote a permutation of $(1, 2, \dots, n\}$. Given two vectors $x, y \in \mathbb{R}^n$ we say that x is Π-lexicolarger than y if, for some index i, $x_{\Pi(i)} > y_{\Pi(i)}$ and $x_{\Pi(j)} \geqslant y_{\Pi(j)}$ for all $j \leqslant i$.

x is a Π-lexicomaximal vector in a set S if x is in S and x is Π-lexicolarger than any $y \neq x$, y in S. x is Π-lexicominimal vector in S if $-x$ is Π-lexicomaximal in the set $-S$.

We will be concerned here with the case where S is the set of optimal solutions to a linear program. In this case we will say that x is a Π-lexicomaximal (minimal) optimal solution.

It is well known that a Π-lexicomaximal (minimal) optimal solution is an extreme point of the feasible domain of the linear program.

Throughout the paper $A_{\cdot j}$ will denote the j-th column of a matrix A. $\lceil a \rceil$ ($\lfloor a \rfloor$) will denote the smallest (largest) integer which is not smaller (larger) than a.

Lemma 2. *Let (A, c) be a pair of an $(m \times n)$ nonnegative integral matrix A and an $(n \times 1)$ nonnegative integral vector c. Let $K = \{k \mid c_k = 1\}$.*

Suppose that for all integer b, $f(b)$, defined by

$$f(b) = \{\min c'x \mid Ax \geqslant b, x \geqslant 0\},$$

is integer. (A is assumed to contain no zero rows so that $f(b)$ is well defined.)

Let Π be a permutation such that $\Pi(i) \in K$ for $i = 1, \dots, |K|$.

Given an integer b define x^0 to be the Π-lexicomaximal optimal solution to

$$\min \quad \{c'x \mid Ax \geqslant b, x \geqslant 0\}.$$

Then x^0 is an extreme point of $\{x \mid Ax \geqslant b, x \geqslant 0\}$ and x_k^0 is integer for all $k \in K$.

Proof. As noted above for any permutation Π, a Π-lexicomaximal optimal solution to a linear program is an extreme solution.

Without loss of generality suppose that $K = \{1, 2, \dots, t\}$ and $\Pi(k) = k$, $k = 1, 2, \dots, t$. Furthermore, assume that for some r, $r \leqslant t$, x_r^0 is not integral, while x_j^0 is integral for $1 \leqslant j \leqslant r-1$.

Define y^0 as follows:

$$y_j^0 = 0, \quad 1 \leqslant j \leqslant r, \qquad y_j^0 = x_j^0, \quad r+1 \leqslant j \leqslant n.$$

y^0 clearly satisfies $Ay^0 \geqslant b'$, $y^0 \geqslant 0$, where

$$b' = b - \sum_{j=1}^{r-1} x_j^0 A_{\cdot j} - \lceil x_r^0 \rceil A_{\cdot r}.$$

Furthermore

$$f(b') \leqslant c'y^0 = c'x^0 - \sum_{j=1}^{r-1} x_j^0 - x_r^0$$

$$< c'x^0 - \sum_{j=1}^{r-1} x_j^0 - \lceil x_r^0 \rceil + 1.$$

Since $f(b')$ as well as

$$c'x^0 - \sum_{j=1}^{r-1} x_j^0 - \lceil x_r^0 \rceil + 1$$

are both integral,

$$f(b') \leqslant c'x^0 - \sum_{j=1}^{r-1} x_j^0 - \lceil x_r^0 \rceil. \tag{2}$$

Let y^* be an optimal solution yielding $f(b')$. Define z^0 by

$$z_j^0 = \begin{cases} x_j^0 + y_j^*, & 1 \leqslant j \leqslant r-1, \\ \lceil x_r^0 \rceil + y_r^*, & j = r, \\ y_j^*, & j > r. \end{cases}$$

z^0 satisfies $Az^0 \geqslant b$ and $z^0 \geqslant 0$. By (2),

$$f(b) \leqslant c'z^0 = f(b') + \sum_{j=1}^{r-1} x_j^0 + \lceil x_r^0 \rceil \leqslant c'x^0 = f(b).$$

Hence $c'z^0 = f(b)$, and z^0 is an optimal solution. But z^0 is Π-lexicolarger than x – thus contradicting the supposition that x_r^0 is not integral.

The above lemma implies the following strengthened version of the pluperfect graph theorem (Theorem 1).

Theorem 3. *Let A be an $(m \times n)$ nonnegative integral matrix with no zero rows. For any integer b define*

$$f(b) = \min\left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0 \right\}.$$

Suppose that $f(b)$ is integer for all integer b. Consider an integer vector b. Let Π be a permutation of $\{1, \dots, n\}$ and let x_{11} be a Π-lexicomaximal *optimal solution to*

$$\min \left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0 \right\}.$$

Then x_{Π} *is an integral extreme point of* $\{x \mid Ax \geqslant b, x \geqslant 0\}$.

Corollary 4. *Let* A *satisfy the conditions of Theorem 3. Given an integer vector* b, *the solution to the integer program*

$$\min \left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0, x \text{ integer} \right\}$$

is unique if and only if the solution to the linear program

$$\min \left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0 \right\}$$

is unique.

Proof. Let x^0 denote the lexicomaximal optimum solution with respect to the identity permutation. If the linear program has more than one optimal solution, there exists an optimal solution y^0 such that $y_r^0 > x_r^0$ for some index r. Let Π be a permutation where $\Pi(1) = r$. The Π-lexicomaximal optimal solution x_{Π} satisfies $x_{\Pi} \neq x^0$.

We note that any matrix A that satisfies the conditions of the above theorem must be a 0–1 matrix. Moreover, A is the node clique incidence matrix of some m-vertex perfect graph, where rows represent the nodes and columns correspond to cliques [1,2]. The problem of finding an integer vector x yielding $f(b)$ can then be viewed as the weighted clique cover problem on a perfect graph. The latter problem is solved polynomially in the seminal paper [4] by Grötschel, Lovász and Schrijver, where the input of the problem consists only of the $(m \times m)$ node vs. node incidence matrix and the vector b.

The algorithm suggested in [4] for solving the weighted clique cover can be schematically described as follows:

0. Set $z_j = 0, j = 1, \ldots, n$.
1. Solve the linear program

$$\min \left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0 \right\},$$

obtaining an optimal solution x^*.

Let $J = \{ j \mid x_j^* > 0\}$.

2. Set $z_j \leftarrow z_j + \lfloor x_j^* \rfloor$ for each $j \in J$. $b \leftarrow b - \sum_{j \in J} \lfloor x_j^* \rfloor A_{.j}$.
3. If $x_j^* - \lfloor x_j^* \rfloor = 0$ for each $j \in J$ stop; z is an optimal integer solution. Otherwise let r be such that $x_r^* - \lfloor x_r^* \rfloor > 0$. Go to 4.
4. Set $z_r \leftarrow z_r + 1$, $b \leftarrow b - A_{.r}$ and go to 1.

Of course the main contribution of [4] is the polynomial algorithm to solve the linear programs defined in Step 1. As mentioned above, in their algorithm A is not listed explicitly, and the optimizing vector x^* will have at most m positive components, i.e., $|J| \leqslant m$.

Grötschel et al. [4] use a direct perfect graph argument to validate the above scheme (in particular the rounding up in Step 4).

A different justification can be obtained by Theorem 3 without referring to the underlying perfect graph. The noninteger vector y, defined by

$$y_j = \begin{cases} x_j^* - \lfloor x_j^* \rfloor, & j \in J, \\ 0, & j \notin J, \end{cases}$$

optimally solves the linear program with $b - \sum_{j \in J} \lfloor x_j^* \rfloor A_{.j}$ as the r.h.s. Thus, if $y_r > 0$, it follows from Theorem 3 that there exists an integer optimal solution to the latter problem with the r-th component being at least 1. This validates Step 4 and the above scheme in general.

The above scheme will (polynomially) generate an optimal integer solution, provided that some polynomial routine (in terms of the length of the input used to define the problem) is used in Step 1. However, such a scheme may not generate an optimal integer solution which is also an extreme point. Theorem 3 suggests the following procedure that finds an optimal integer extreme solution.

0. Set $z_j = 0, j = 1, \ldots, n$.
1. Solve the linear program

$$f(b) = \min \left\{ \sum_{j=1}^{n} x_j \mid Ax \geqslant b, x \geqslant 0 \right\}.$$

Let r be some positive component of an optimal solution. (If none exists, z is an optimal integer extreme solution).

2. Find the largest integer, p, in $[0, \bar{b}]$, where $\bar{b} = \max\{b_i\}$, such that

$$f(b) = f(b - pA_{.r}) + p.$$

3. Set $z_r = p$, $b \leftarrow b - pA_{.r}$.
4. Go to 1.

Theorem 3, coupled with the following observation, justifies the validity of this algorithm. Let $z_{i(j)}$, $j = 1,\ldots,q$, be the positive component of z produced at the jth iteration of the algorithm. Then, z is a Π-lexicomaximal optimum solution for any permutation Π such that $\Pi(j) = i(j)$, $j = 1,\ldots,q$.

The integer p defined in Step 2, can be obtained by applying a binary search on the integers in $[0, \bar{b}]$. Therefore, p is computable by solving $O(\log \bar{b})$ linear programs like those defined in Step 1. The algorithm iterates at most m times since an extreme point has at most m positive components.

Thus, we conclude that the above procedure will generate an optimal integer extreme point provided that some polynomial routine is available for solving the linear program in Step 1. (In fact this routine will need to produce only the value $f(b)$ and the index of one positive component of an optimal solution.)

We now demonstrate that the linear program in Theorem 3 can have a noninteger extreme optimal solution even when b is the vector all of whose components are equal to 1.

Example. Let G be the six vertex graph shown in Figure 1. G is a perfect graph since its complement consists of three isolated edges only. The node clique incidence matrix of G is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

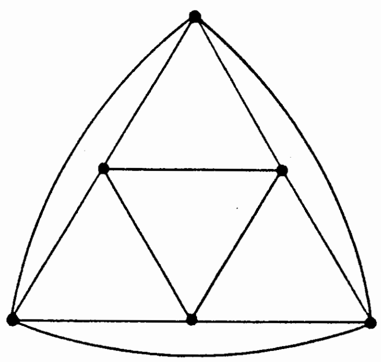

Fig. 1.

If b is the vector all of whose components are equal to 1, $f(b)$ in Theorem 3 is also attained at the noninteger optimal extreme point $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$.

We conclude by noting that the above extremality property of integral solutions is also shared by covering and packing problems defined by balanced matrices. We state these results but omit the proofs since they are very similar to that of Lemma 2. These results strengthen the results in [3]. Recall that a 0–1 matrix is balanced if it does not contain a square submatrix of odd order whose column and row sums are all equal to 2.

Theorem 5. *Let A be an $(m \times n)$ balanced matrix, and let b and d be integer vectors. Let Π be a permutation of $\{1,\ldots,n\}$.*

(1) *Every Π-lexicomaximal optimal solution to the linear program*

$$\min\left\{\sum_{j=1}^{n} x_j \,\middle|\, Ax \geqslant b,\ 0 \leqslant x \leqslant d\right\}$$

is integral and an extreme point of $\{x \mid Ax \geqslant b, 0 \leqslant x \leqslant d\}$.

(2) *Every Π-lexicominimal optimal solution to the linear program*

$$\max\left\{\sum_{j=1}^{n} x_j \,\middle|\, Ax \leqslant b,\ 0 \leqslant x \leqslant d\right\}$$

is integral and an extreme point of $\{x \mid Ax \leqslant b, 0 \leqslant x \leqslant d\}$.

References

[1] D.R. Fulkerson, "Blocking and antiblocking pairs of polyhedra", *Math. Programming* **1**, 168–194 (1971).

[2] D.R. Fulkerson, "Antiblocking polyhedra", *J. Comb. Theory* **12**, 50–71 (1972).

[3] D.R. Fulkerson, A.J. Hoffman and R. Oppenheim, "On balanced matrices", *Math. Programming Study* **1**, 120–132 (1974).

[4] M. Grötschel, L. Lovász and A. Schrijver, "The ellipsoid method and its consequences in combinatorial optimization", *Combinatoria* **1**, 169–197 (1981).