Maximum Coverage with Balls of Different Radii

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Abstract

We consider the problem of maximum weighted coverage of points by a set of balls of different radii in the Euclidean space R^d. We observe that the problem can be formulated as a multiple choice coverage problem. Therefore, the approximation results in [1] are applicable, and for any fixed d there is a polynomial algorithm which produces an approximation within a factor of $1 - 1/e$.

Let $P = \{v_1, ..., v_n\}$ be a set of n points in R^d, where $n \geq d$. Each point v_i, $i = 1, ..., n$, is associated with a nonnegative weight w_i. Let $\{r_1, ..., r_p\}$ be a set of radii, nonnegative real numbers. For each pair of points $x, y \in R^d$, let $d(x, y)$ denote the Euclidean distance between them. For $j = 1, ..., p$, and $x \in R^d$ let $D_j(x)$ denote the Euclidean ball of radius r_j centered at x, i.e., $D_j(x) = \{y \in R^d : d(x, y) \leq r_j\}$.

The *coverage problem* is to maximize the total weight of points that can be covered by p balls of radii $\{r_1, ..., r_p\}$. Formally, the problem is to find p points, $x_1, ..., x_p$, in R^d such that the total weight of the subset of points in P contained in $\cup_{j=1}^{p} D_j(x_j)$, is maximized. The problem has recently been shown to be NP-hard even in the one-dimensional case, [5]. Note that the latter case corresponds to the problem of covering points on the real line by line segments of different lengths.

When the balls are of the same radius, $r_j = r$ for $j = 1, ..., p$, the problem is NP-hard for $d \geq 2$, [4]. This equal radii case reduces to the maximum coverage problem, and therefore can be approximated within a factor of $1 - (1 - 1/p)^p$ of the optimum by a greedy algorithm. See [2, 3, 6]. (The greedy algorithm can be implemented in $O(pn^2)$ time in R^2.)

We will show that for different radii, the above problem can be formulated as a multiple choice coverage problem, and therefore the approximation results in [1] are applicable.

For each $j = 1, ..., p$, let $F_j = \{S_j^1, ..., S_j^m\}$ be the subcollection of all maximal subsets of points in P that can be covered by a ball of radius r_j. (Note that since the radius of the ball is known $m = O(n^d)$.)

For $t = 1, ..., m$, define $a(i; j, t) = 1$ if $v_i \in S_j^t$, and $a(i; j, t) = 0$ if $v_i \notin S_j^t$.

The formulation is then:

$\max \sum_{i=1}^{n} w_i z_i$

subject to,

$\sum_{j=1}^{p} \sum_{t=1}^{m} a(i; j, t) x_{j,t} \geq z_i$, $i = 1, ..., n$,

$\sum_{t=1}^{m} x_{j,t} = 1$, $j = 1, ..., p$,

$x_{j,t} \in \{0, 1\}$, $j = 1, ..., p$; $t = 1, ..., m$,

$0 \leq z_i \leq 1$, $i = 1, ..., n$.

For each $i = 1, ..., n$, $\sum_{j=1}^{p} \sum_{t=1}^{m} a(i; j, t)$ is bounded above by $k = O(pn^d)$. From the results in [1], (see Remark 1), we now conclude that when d is fixed there is a polynomial algorithm which produces an approximation within a factor of $1-(1-1/k)^k$ of an optimal solution.

we note that the above scheme is applicable to any metric defined on R^d, provided that the respective collections F_j, $j=1,...,p$, are of polynomial cardinality. The rectilinear norm is one such example.

In the above coverage problem the p balls can be centered anywhere in R^d. Consider a version of the model where the centers of the balls are retricted to some prespecified discrete subset $Q=\{u_1,...,u_q\}\subset R^d$. For $t=1,...,q$, define $a(i;j,t)=1$ if $v_i\in D_j(u_t)$, and $a(i;j,t)=0$ if $v_i\notin D_j(u_t)$. We obtain the following formulation:

$\max \sum_{i=1}^n w_i z_i$

subject to,

$\sum_{j=1}^p \sum_{t=1}^q a(i;j,t)x_{j,t}\geq z_i$, $i=1,...,n$,

$\sum_{t=1}^q x_{j,t}=1$, $j=1,...,p$,

$x_{j,t}\in\{0,1\}$, $j=1,...,p$; $t=1,...,q$,

$0\leq z_i\leq 1$, $i=1,...,n$.

From the results in [1] we conclude that there is a polynomial algorithm for the discrete version which produces an approximation within a factor of $1-(1-1/(pq))^{pq}$ of an optimal solution.

As a final remark we note that the results in [1] are set theoretic. An open question is whether approximation factors larger than $1-1/e$ are achievable by utilizing the geometric properties of the problem.

References

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