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SHORT COMMUNICATION

ON A CHARACTERIZATION OF P-MATRICES

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We consider here the linear complementarity problem which is to find vectors $\omega \in \mathrm{R}^n$, $Z \in \mathrm{R}^n$ satisfying

$$\omega = MZ + g\,, \quad \omega \geq 0\,, \quad Z \geq 0\,, \quad \omega^{\mathrm{T}} Z = 0\,, \tag{1}$$

where ω^{T} denotes the transpose of ω, and M, g are given $n \times n$ and $n \times 1$ matrices, respectively. If A is a matrix, we will denote by $A_{.j}$ the j^{th} column of A and by $A_{i.}$ the i^{th} row of A. M is said to be a Q-matrix if (1) has a solution for every $g \in \mathrm{R}^n$. M is said to be a P-matrix if all its principal minors are strictly positive. It was shown in [1; 3; 4; 6] that M is a P-matrix if and only if (1) has a unique solution for every $g \in \mathrm{R}^n$.

Murty [5] refined the above characterization by proving that M is a P-matrix if and only if (1) has a unique solution for every $g \in \Gamma$.

$$\Gamma = \{I_{.1}, ..., I_{.n}, -I_{.1}, ..., -I_{.n}, M_{.1}, ..., M_{.n}, -M_{.1}, ..., -M_{.n}, e\},$$

where $I_{.j}$ is the j^{th} column of the identity matrix of order $n \times n$ and $e = (1, ..., 1)^{\mathrm{T}}$.

We improve this result and show that M is a P-matrix if and only if (1) has a unique solution whenever $g \in \Gamma_1$, where

$$\Gamma_1 = \{I_{.1}, ..., I_{.n}, M_{.1}, ..., M_{.n}, -M_{.1}, ..., -M_{.n}, e\}\,.$$

The main result

Lemma 1. *If* (1) *has a unique solution whenever* $g \in \{M_{.1}, \ldots, M_{.n}\}$, *then* $\omega = Z = 0$ *is the unique complementary solution corresponding to* $g = 0$.

Proof. Suppose that this is not the case. Then without loss of generality we assume that (ω^*, Z^*) is a solution to (1) corresponding to $g = 0$, where $\omega^* = (0, \ldots, 0, \omega^*_{k+1}, \ldots, \omega^*_n)^{\mathrm{T}}$, $Z^* = (Z^*_1, \ldots, Z^*_k, 0, \ldots, 0)^{\mathrm{T}}$, $Z^*_i > 0$, $1 \leq i \leq k$, and $\omega^*_i \geq 0$, $k + 1 \leq i \leq n$. We can further assume that $Z^*_1 = 1$.

Consider now the complementarity problem (1) corresponding to $g = M_{.1}$. It is easy to see that the following are two different complementary feasible solutions to this problem:

$$(\omega^1; Z^1) = (0, \ldots, 0, \omega^*_{k+1}, \ldots, \omega^*_n; 0, Z^*_2, \ldots, Z^*_k, 0, \ldots, 0)\,,$$
$$(\omega^2; Z^2) = (0, \ldots, 0, 2\omega^*_{k+1}, \ldots, 2\omega^*_n; 1, 2Z^*_2, \ldots, 2Z^*_k, 0, \ldots, 0)\,.$$

This contradicts the uniqueness of a complementary solution to (1) when $g = M_{.1}$, hence the theorem follows.

Theorem 1. *If* (1) *has a unique solution whenever* $g \in \{M_{.1}, \ldots, M_{.n}, e\}$, *where* $e = (1, \ldots, 1)^{\mathrm{T}}$, *then M is a* Q-*matrix.*

Proof. For every $Z \geq 0$, let $I_+(Z)$ and $I_0(Z)$ denote the sets of indices corresponding to the positive and zero components of Z, i.e., $I_+(Z) = \{i\colon Z_i > 0\}$ and $I_0(Z) = \{i\colon Z_i = 0\}$. Lemma 1 and the fact that (1) has a unique solution when $g = e$ imply that the system

$$\begin{array}{lll} M_{i.}Z + t = 0 & \text{for} \quad i \in I_+(Z)\,, & \\ & & 0 \neq Z \geq 0\,, \quad t \in \{0, 1\} \\ M_{i.}Z + t \geq 0 & \text{for} \quad i \in I_0(Z)\,, & \end{array}$$

is inconsistent. Following Karamardian [2], we obtain the result that M is regular and therefore is a Q-matrix.

Result 1 [4, 4.9, 4.10]. *If M is a* Q-*matrix and* (1) *has a unique solution corresponding to each* $g \in \{I_{.1}, \ldots, I_{.n}, -M_{.1}, \ldots, -M_{.n}\}$, *then M is a* P-*matrix.*

We now state the main result of this work.

Theorem 2. *M is a* P-*matrix if and only if* (1) *has a unique solution for each* $g \in \Gamma_1$, *where*

$$\Gamma_1 = \{I_{.1}, ..., I_{.n}, M_{.1}, ..., M_{.n}, -M_{.1}, ..., -M_{.n}, e\} ,$$

with $e = (1, ..., 1)^{\mathrm{T}}$.

Proof. The necessity of the conditions is a well-known result as mentioned above, while their sufficiency follows from Theorem 1 and Result 1.

References

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