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Locating service centers with precedence constraints

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Abstract

Isotonic regression models have been used extensively with various objective functions. In this paper we consider the isotonic regression model with the minmax criterion. The model is presented and interpreted as a multi-center location problem on the real line, with precedence constraints on the locations of the centers. Viewed as a location problem, this model generalizes the classical (weighted) 1-center problem on the line. We obtain an explicit expression for the optimal objective value, and use it to develop an efficient subquadratic algorithm for solving the problem. Finally we present and discuss a generalization which unifies the above model and other location problems with distance constraints.

Introduction

Consider a set of n demand points (customers), $v_1, v_2, \ldots, v_n$, on the real line. Each customer v_i, $i = 1, \ldots, n$, has to locate a service center, represented by a point x_i on the line. x_i will serve v_i exclusively, and the nature of the service is such that v_i prefers x_i to be as near to it as possible. Therefore, with no constraints on the locations of $x_1, \ldots, x_n$, each x_i will be established at v_i. We assume that there exists a feasible set S, and each vector of locations $(x_1, \ldots, x_n)$ must be in S. In particular, we focus on the case where S reflects precedence constraints on the locations of the n centers. Such location models have been discussed in the literature under the title of isotonic regression [2]. Several objective functions have been considered. For example, [4, 13, 22, 25] consider the isotonic mean regression where the objective is to minimize the weighted sum of squared distances between the customers and their respective centers. The isotonic median regression model has been studied in [3, 5, 6, 18–21]. The objective there is to minimize the weighted sum of distances (absolute values) between the demand points and their respective centers.

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In this work we concentrate on the bottleneck criterion and wish to minimize the maximum weighted distance. To facilitate the discussion, suppose that $w_1, \ldots, w_n$ are positive weights, and consider the following model:

$$\begin{aligned} &\underset{(x_1,\ldots,x_n)}{\text{Minimize}} \quad \underset{1\le i\le n}{\text{Maximum}} \{w_i |x_i - v_i|\}, \\ &\text{subject to} \quad (x_1,\ldots,x_n) \in S. \end{aligned} \tag{1}$$

The set S is defined by precedence constraints. Specifically, given is a directed graph $G=(V,E)$, where $V=\{\bar{v}_1,\ldots,\bar{v}_n\}$. An arc in E is an ordered pair $(\bar{v}_i,\bar{v}_j)$, $i \neq j$. (In general E is allowed to contain both $(\bar{v}_i,\bar{v}_j)$ and $(\bar{v}_j,\bar{v}_i)$.) S is now defined by

$$S=\{(x_1,\ldots,x_n)\,|\,\text{for all } 1\le i,j\le n,\ x_i \le x_j \text{ if } (\bar{v}_i,\bar{v}_j)\in E\}. \tag{2}$$

The constraints in S are also recognized as variable upper bound constraints [23]. The isotonic regression model (1)–(2) generalizes the classical weighted 1-center problem studied in location theory [12]. In the latter the objective is to select a single point (center) x on the line that will minimize the maximum weighted distance,

$$\underset{x}{\text{Minimize}} \quad \underset{1\le i\le n}{\text{Maximum}} \{w_i|x-v_i|\}. \tag{3}$$

To obtain (3) from (1) set G to be a directed cycle. That would imply $x_1 = x_2 = \cdots = x_n$. (3) can be formulated as a 2-variable linear program,

$$\begin{aligned} z^* = \text{Minimum} \quad & z, \\ \text{subject to} \quad & w_i(x-v_i) \le z, \\ & -w_i(x-v_i) \le z, \quad i=1,\ldots,n. \end{aligned} \tag{4}$$

Using the linear programming techniques in [16], the optimal solution to (4), (x^*, z^*), can be computed in $O(n)$ time. It is known that this model satisfies an interesting separability property. For each pair, i, j let z^*_{ij} be the solution value of the 1-center problem restricted to the points v_i and v_j:

$$\begin{aligned} z^*_{ij} = \text{Minimum} \quad & z_{ij}, \\ \text{subject to} \quad & w_i(x-v_i) \le z_{ij}, \\ & -w_i(x-v_i) \le z_{ij}, \\ & w_j(x-v_j) \le z_{ij}, \\ & -w_j(x-v_j) \le z_{ij}. \end{aligned} \tag{5}$$

Then z^*, the optimal solution value, is given by $z^* = \text{Maximum}_{1\le i,j\le n}\{z^*_{ij}\}$.

Since $z_{ij}^* = |v_i - v_j|/(1/w_i + 1/w_j)$,

$$z^* = \underset{1 \le i, j \le n}{\text{Maximum}} \left\{ \frac{|v_i - v_j|}{1/w_i + 1/w_j} \right\}. \tag{6}$$

We conclude that the $2n$-constraint linear program (4) is decomposed into $n(n-1)/2$ 4-constraint linear programs. Our goal here is two-fold. First, we will show that the general model defined by (1)–(2) possesses the same separability property, which implies an explicit expression for its optimal value. Second, we will use that expression to derive an efficient algorithm to locate $x_1, \ldots, x_n$.

We formulate (1)–(2) as an $(n+1)$-variable program,

$$\begin{aligned} &\text{Minimize} \quad z, \\ &\text{subject to} \quad w_i(x_i - v_i) \le z, \\ &\qquad -w_i(x_i - v_i) \le z, \quad i = 1, \ldots, n, \\ &\qquad x_i \le x_j \quad \text{if } (\bar{v}_i, \bar{v}_j) \in E,\ 1 \le i, j \le n. \end{aligned} \tag{7}$$

We note in passing that (7) can be solved efficiently as a parametric linear flow problem. However, the algorithms that we will develop have a lower complexity bound.

The algorithms

Given the precedence constraint graph $G = (V, E)$, $V = \{\bar{v}_1, \ldots, \bar{v}_n\}$, we define its transitive closure $G^* = (V, E^*)$, by setting

$$E^* = \{(\bar{v}_i, \bar{v}_j) \mid \text{there exists a directed path in } G \text{ from } \bar{v}_i \text{ to } \bar{v}_j.\}$$

For each pair i, j, $1 \le i, j \le n$, with $(\bar{v}_i, \bar{v}_j) \in E^*$ define

$$\begin{aligned} z_{ij}^* = \text{Minimum} \quad & z_{ij}, \\ \text{subject to} \quad & w_i(x_i - v_i) \le z_{ij}, \\ & -w_i(x_i - v_i) \le z_{ij}, \\ & w_j(x_j - v_j) \le z_{ij}, \\ & -w_j(x_j - v_j) \le z_{ij}, \\ & x_i \le x_j. \end{aligned} \tag{8}$$

It is easily observed that

$$z_{ij}^* = \begin{cases} 0, & \text{if } v_i \le v_j, \\ \dfrac{v_i - v_j}{1/w_i + 1/w_j}, & \text{otherwise.} \end{cases} \tag{9}$$

Define

$$\bar{z} = \underset{(\bar{v}_i, \bar{v}_j) \in E^*}{\text{Maximum}} \{z_{ij}^*\}. \tag{9a}$$

Let z^* be the optimal value of (7). We claim that $z^* = \bar{z}$. Reformulate (7) as

$$\begin{aligned} z^* = \text{Minimum} \quad & z, \\ \text{subject to} \quad & w_i(x_i - v_i) \le z, \\ & -w_i(x_i - v_i) \le z, \quad i = 1, \dots, n, \\ & x_i \le x_j \text{ if } (\bar{v}_i, \bar{v}_j) \in E^*, \ 1 \le i, j \le n. \end{aligned} \tag{10}$$

Lemma 1. $\bar{z} \le z^*$.

Proof. Let $(x_1^*, \dots, x_n^*)$ and z^* be an optimal solution to (10). Then, for each i, j with $(\bar{v}_i, \bar{v}_j) \in E^*$, (x_i^*, x_j^*, z^*) constitute a feasible solution to (8). Therefore, $z_{ij}^* \le z^*$. Thus, $\bar{z} \le z^*$. □

To prove that $\bar{z} \ge z^*$, we constructively generate a feasible solution to (10) with $z = \bar{z}$.

Algorithm 1.

Step 1. If $\bar{z} = 0$, set $x_i = v_i$ for $i = 1, \dots, n$ and stop.

Step 2. (a) For $i = 1, \dots, n$, define $c_i = v_i - \bar{z}/w_i$.
(b) Sort the numbers $\{c_i\}$, $i = 1, \dots, n$, into a list X.
(c) Define all elements i, $i = 1, \dots, n$, to be unlabelled.

Step 3. Consider the largest entry in X, say c_i. If i is labelled, delete c_i from X and go to Step 5. Otherwise, go to Step 4.

Step 4. (a) Label i, and set $x_i = c_i$.
(b) For each unlabelled j with $(\bar{v}_i, \bar{v}_j) \in E^*$ set $x_j = x_i$ and label j.

Step 5. If X is nonempty return to Step 3. Otherwise stop.

Proposition 2. *Let* $(x_1, \dots, x_n)$ *be defined by Algorithm* 1. *Then* $(x_1, \dots, x_n)$ *and* $\bar{z}$ *constitute a feasible solution to* (10).

Proof. It is sufficient to consider the case when $\bar{z} > 0$. We will first show that if $(\bar{v}_i, \bar{v}_j)$, is in E^* then $x_i \le x_j$. Consider the iteration when x_i is assigned its value in Step 4. If x_j

was also assigned a value in this iteration, (Step 4(b)), then $x_i = x_j$. Otherwise, x_j had been defined in an earlier iteration. But since the values assigned to the variables by the algorithm form a nonincreasing sequence we must have had $x_j \geq x_i$.

Next we prove that $w_i|x_i - v_i| \leq \bar{z}$ for $i = 1,\dots,n$. If x_i is defined in Step 4(a), then $w_i|x_i - v_i| = w_i|v_i - \bar{z}/w_i - v_i| = \bar{z}$. Otherwise, there exists some k, $1 \leq k \leq n$, such that $(\bar{v}_k, \bar{v}_i) \in E^*$, and x_k and x_i were defined in Step 4(a) and Step 4(b), respectively, of the same iteration, i.e., $x_k = x_i = c_k$.

Consider two subcases:

(I) $x_i < v_i$.

(II) $x_i \geq v_i$.

Since x_k and x_i are labelled in the same iteration, and since x_k is defined in Step 4(a), we have $c_i \leq c_k$. When subcase (I) holds, $x_i < v_i$, and therefore,

$$\begin{aligned} w_i|v_i - x_i| &= w_i(v_i - x_i) = w_i(v_i - c_k) \\ &\leq w_i(v_i - c_i) = w_i\left(v_i - v_i + \frac{\bar{z}}{w_i}\right) = \bar{z}. \end{aligned}$$

Finally suppose that (II) holds. Then,

$$\begin{aligned} w_i|v_i - x_i| &= w_i(x_i - v_i) = w_i(c_k - v_i) = w_i\left(v_k - \frac{\bar{z}}{w_k} - v_i\right) \\ &= w_i\left(\frac{v_k - v_i}{1/w_k + 1/w_i}\left(\frac{1}{w_k} + \frac{1}{w_i}\right) - \frac{\bar{z}}{w_k}\right) \\ &\leq w_i\left(z_{ki}^*\left(\frac{1}{w_k} + \frac{1}{w_i}\right) - \frac{\bar{z}}{w_k}\right), \end{aligned}$$

where the last equality follows from (9). Since, by definition, $\bar{z} \geq z_{ki}^*$, we obtain

$$w_i|v_i - x_i| \leq w_i\left(\bar{z}\left(\frac{1}{w_k} + \frac{1}{w_i}\right) - \frac{\bar{z}}{w_k}\right) = \bar{z},$$

and the proof is complete. □

The above lemma and proposition imply the following result.

Theorem 3. *The optimal solution value to* (10) *is given by* $\bar{z}$, *defined by* (9a), *and an optimal set of locations* $(x_1,\dots,x_n)$ *is defined by Step* 1 *or* 4 *of Algorithm* 1.

To analyze the complexity of Algorithm 1 we first note that it requires, as input, E^*, the arc set of the transitive closure graph $G^* = (V, E^*)$, as well as the value of $\bar{z}$. When E^* is given, $\bar{z}$ is computed in $O(|E^*|)$ time, and the algorithm is easily observed to consume $O(n\log n + |E^*|)$ time.

There are at least two different approaches to construct E^*. The first is by applying a depth first search from each node of the precedence constraint graph $G = (V, E)$ to find all nodes which are reachable from that node. The total effort involved is $O(n|E|)$. The second approach is to use matrix multiplication techniques, as explained in [1], to find E^*. If we use the technique given in [8], which has the lowest complexity bound to date, E^* is produced in $O(n^{2.376})$ time.

We conclude that the total time to solve the location model (1)–(2) with the first approach is $O(n|E| + |E^*|) = O(n|E|)$. The respective complexity using the second method is $O(n^{2.376})$.

The bottleneck term in the complexity bound of Algorithm 1 comes from the effort to compute the transitive closure graph $G^* = (V, E^*)$. We will now present a more elaborate algorithm, Algorithm 2, which solves (1)–(2) without explicitly generating G^*. The complexity of Algorithm 2 is only $O(n \log^2 n + |E| \log n)$.

Algorithm 2 is based on an efficient search in a well-structured set which is known to contain z^*. (It avoids the computation of z^*, as suggested by Theorem 3, since that will require knowing E^* explicitly.) Specifically, we consider the set R,

$$R = \left\{ \frac{v_i - v_j}{1/w_i + 1/w_j} \middle| \, 1 \le i, j \le n \right\}.$$

From Theorem 3 it follows that z^* is indeed an element of R. Thus, z^* can be characterized as the smallest element z in R, such that there exists a set of locations $(x_1, \ldots, x_n)$, which together with z constitute a feasible solution to (7). An element z for which there is such a set $(x_1, \ldots, x_n)$ will be called a feasible element of R. z^* is therefore the smallest feasible element.

Given an element $z > 0$ in R, we first develop a feasibility test. For each i, $i = 1, \ldots, n$, define $a_i = v_i - z/w_i$, and $b_i = v_i + z/w_i$. z is feasible if and only if the following system is feasible,

$$\begin{aligned} &a_i \le x_i \le b_i, \quad i = 1, \ldots, n; \\ &x_i \le x_j \quad \text{if } (\bar{v}_i, \bar{v}_j) \in E, \ 1 \le i, j \le n. \end{aligned} \tag{11}$$

Each constraint $x_i - x_j \le 0$ in (11) has at most one positive coefficient and at most one negative coefficient. Therefore, it follows from [9, 24], that if (11) is feasible then it has a least element $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$: $\bar{x}$ is a feasible solution for (11), and every other feasible solution $x = (x_1, \ldots, x_n)$ satisfies

$$\bar{x} \le x.$$

(Although we do not use this property, we note that (11) also has a largest element.) To test the feasibility of (11) consider the relaxed system

$$\begin{aligned} &x_i \ge a_i, \quad i = 1, \ldots, n; \\ &x_i \le x_j \quad \text{if } (\bar{v}_i, \bar{v}_j) \in E, \ 1 \le i, j \le n. \end{aligned} \tag{12}$$

Since (12) is feasible it has a least element, say $\bar{x}$. Thus, (11) is feasible if and only if $\bar{x}_i \le b_i$, $i = 1, \dots, n$. To summarize, given a positive element z in R we test its feasibility by computing the least element of (12) and comparing it with $(b_1, \dots, b_n)$.

The feasibility test (Computing the least element).

Step 1. Sort the numbers $\{a_i\}$, $i = 1, \dots, n$, into a list X.

Step 2. Define all elements, $i = 1, \dots, n$, to be unlabelled.

Step 3. Consider the largest entry in X, say a_i. If i is labelled, delete a_i from X and go to Step 5. Otherwise, proceed to Step 4.

Step 4. (a) Starting at node $\bar{v}_i$ of G, perform a depth first search on G to find $V(i) = \{\bar{v}_j \mid \text{there exists a directed path from } \bar{v}_i \text{ to } \bar{v}_j \text{ consisting only of nodes with unlabelled indices}\}$.

(b) Set $x_i = a_i$ and $x_j = x_i$ for all j such that $\bar{v}_j \in V(i)$. Label i and all j such that $\bar{v}_j \in V(i)$.

Step 5. If X is nonempty return to Step 3. Otherwise stop.

The complexity of the feasibility test is easily observed to be $O(n \log n + |E|)$. To validate the feasibility test we need the following lemma.

Lemma 4. *Consider the set $V(i)$, defined in Step 4 of the feasibility test. If $(\bar{v}_i, \bar{v}_j) \in E^*$ and $\bar{v}_j \notin V(i)$, then j has already been labelled.*

Proof. Suppose that j has not been labelled. From the fact that $(\bar{v}_i, \bar{v}_j) \in E^*$ but $\bar{v}_j \notin V(i)$, it follows that there exist indices k and u, such that k is labelled, u is not, and $(\bar{v}_k, \bar{v}_u) \in E$. However, this is not possible since u should have been labelled together with k in the same iteration. □

Proposition 5. *The feasibility test finds the least element of the linear system* (12).

Proof. For each iteration t of the feasibility test let $i(t)$ be the index selected at Step 3. It will suffice to prove by induction on t, that for each j in $V(i(t))$ (defined in Step 4), the value $a_{i(t)}$ assigned to x_j in Step 4 is a lower bound on this variable for any feasible solution to (12), and $x_k \ge a_{i(t)}$ for each k with $(\bar{v}_j, \bar{v}_k) \in E^*$ and any feasible solution x to (12).

Since $(\bar{v}_{i(t)}, \bar{v}_j) \in E^*$ we must have $x_j \ge x_{i(t)} \ge a_{i(t)}$ for any feasible solution x to (12). Next consider some k with $(\bar{v}_j, \bar{v}_k) \in E^*$. If $\bar{v}_k \in V(i(t))$, then x_k is also assigned the value $a_{i(t)}$, so certainly $x_k \ge a_{i(t)}$ for any feasible solution. Otherwise, since $(\bar{v}_j, \bar{v}_k) \in E^*$ it follows from Lemma 4 that there exists an index u, labelled in Step 3 of an earlier iteration and $(\bar{v}_u, \bar{v}_k) \in E^*$. By the induction hypothesis $x_k \ge a_u$ for any feasible solution to (12). Since $a_u \ge a_{i(t)}$ we obtain $x_k \ge a_{i(t)}$ and the proof is complete. □

Having established a feasibility test for an element z in R, we now present our procedure, Algorithm 2, to find z^*, the smallest feasible element in R. We directly apply the sophisticated search procedure described in the appendix of [17] with the further improvement suggested in [7]. Since each feasibility test consumes $O(n\log n + |E|)$ time, it follows from [7, 17], that the total effort to find z^* is $O(n\log^2 n + |E|\log n)$. Finally, given the value of z^*, we note that a set of optimal locations $(x_1^*, \ldots, x_n^*)$ can be generated by the feasibility test with $a_i = v_i - z^*/w_i$, $i = 1, \ldots, n$.

To conclude, we consider the unweighted case of (1)–(2), where $w_i = 1$, $i = 1, \ldots, n$. In this case z^* is given by

$$\begin{aligned} z^* = \text{Minimum} \quad & z, \\ \text{subject to} \quad & v_i - z \le x_i \le v_i + z, \quad i = 1, \ldots, n, \\ & x_i \le x_j \text{ if } (\bar{v}_i, \bar{v}_j) \in E,\ 1 \le i, j \le n. \end{aligned}$$

Substitute $y_i = x_i + z$, $i = 1, \ldots, n$. Let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$ be the least element of the system

$$\begin{aligned} & y_i \ge v_i, \quad i = 1, \ldots, n; \\ & y_i \le y_j \quad \text{if } (\bar{v}_i, \bar{v}_j) \in E,\ 1 \le i, j \le n. \end{aligned} \tag{12a}$$

Then for any positive z, the vector $(\bar{y}_1 - z, \ldots, \bar{y}_n - z)$ is a least element of the system

$$\begin{aligned} & x_i \ge v_i - z, \quad i = 1, \ldots, n; \\ & x_i \le x_j \quad \text{if } (\bar{v}_i, \bar{v}_j) \in E,\ 1 \le i, j \le n. \end{aligned}$$

Using the above discussion

$$\begin{aligned} z^* = \text{Minimum} \quad & z, \\ \text{subject to} \quad & \bar{y}_i - z \le v_i + z, \quad i = 1, \ldots, n. \end{aligned}$$

Therefore, $z^* = \text{Maximum}\{(\bar{y}_i - v_i)/2 \mid i = 1, \ldots, n\}$, and the unweighted version of (1)–(2) can be solved in $O(n\log n + |E|)$ time, as follows:

(a) Compute $\bar{y}$, the least element of the system (12a).
(b) Calculate $z^* = \text{Maximum}\{(\bar{y}_i - v_i)/2 \mid i = 1, \ldots, n\}$.
(c) Calculate the optimal solution x^* by $x_i^* = \bar{y}_i - z^*$, $i = 1, \ldots, n$.

Generalizations and concluding comments

We have focused in this paper on the isotonic regression model with the criterion of minimizing the maximum weighted distance. However, we note that some of the results can be extended to more general forms of the set S in (2). Suppose, for example, that each directed arc $(\bar{v}_i, \bar{v}_j)$ of the precedence constraint graph $G = (V, E)$ is associated with a real number a_{ij}. Define S by

$$S = \{(x_1, \ldots, x_n) \mid \text{for all } 1 \le i, j \le n,\ x_i \le x_j + a_{ij} \text{ if } (\bar{v}_i, \bar{v}_j) \in E\}. \tag{13}$$

This definition unifies the isotonic regression model, i.e., $a_{ij} = 0$ for all $(\bar{v}_i, \bar{v}_j) \in E$, as well as the distance constraint systems $(|x_i - x_j| \leq a_{ij})$ discussed in [10]. In our definition a_{ij} can be of any sign. Also, unlike [10], we allow $a_{ij} \neq a_{ji}$ if both $(\bar{v}_i, \bar{v}_j)$ and $(\bar{v}_j, \bar{v}_i)$ are in E. The result stated in Theorem 3 can be extended to the generalized model. Consider the transitive closure graph $G^* = (V, E^*)$ defined above. For each directed arc $(\bar{v}_i, \bar{v}_j) \in E^*$, let d_{ij} denote the length of a shortest directed path from $\bar{v}_i$ to $\bar{v}_j$ in G with respect to the arc weights $\{a_{ij}\}$. (If S in (13) is nonempty the graph G has no negative cycles. Therefore, the "distances" d_{ij} are well defined.)

For each pair i, j with $(\bar{v}_i, \bar{v}_j) \in E^*$ define

$$z^*_{ij} = \begin{cases} 0, & \text{if } v_i \leq v_j + d_{ij}, \\ \dfrac{v_i - v_j - d_{ij}}{1/w_i + 1/w_j}, & \text{otherwise.} \end{cases} \tag{14}$$

We now state without a proof the generalization of Theorem 3 above.

Theorem 6. *Let z^* be the optimal value of the generalized model defined by* (1) *and* (13). *Then*

$$z^* = \underset{(\bar{v}_i, \bar{v}_j) \in E^*}{\text{Maximum}} \{z^*_{ij}\}. \tag{15}$$

The generalized model can be solved in strongly polynomial time by computing z^* from (15). The complexity of this approach is dominated by the effort to compute the "distances" $\{d_{ij}\}$. If we use the algorithm in [11], the time is $O(n|E| + n^2 \log n)$. We do not know how to improve upon this bound, by extending the approach of Algorithm 2 above. The difficulty lies in explicitly identifying an equivalent of the set R above which will contain z^* and at the same time enable an efficient search.

At this point we do not know how to extend the explicit representation of z^* in (14)–(15) to incorporate a more general feasible set S. However, we note that a strongly polynomial time algorithm exists for the following generalization of the model.

Suppose that S is defined by a given system of linear inequalities in the variables $(x_1, \ldots, x_n)$, where no inequality contains more than two variables. The feasibility of such a system can be verified in strongly polynomial time by the algorithm in [15]. Thus, optimizing (1) over such a system can also be done in strongly polynomial time using the novel binary search idea in [14].

The isotonic regression model (1)–(2) and Algorithm 1 can be easily extended to the multi-valued case. Multi-valued cases are also discussed in the isotonic mean regression and isotonic median regression models mentioned in the introduction. We briefly define this model for our minmax criterion.

In the multi-valued case each service center, i, should serve $t(i)$ customers located on the real line at $v_{i,1}, \ldots, v_{i,t(i)}$. The bottleneck criterion problem is:

$$\begin{array}{ll} \underset{(x_1,\ldots,x_n)}{\text{Minimize}} & \underset{\substack{1\le i\le n\\ 1\le j\le t(i)}}{\text{Maximum}}\{w_{i,j}|x_i - v_{i,j}|\}, \\ \text{subject to} & (x_1,\ldots,x_n)\in S, \end{array} \tag{16}$$

where S is defined in (2).

Algorithms 1 and 2, and the treatment suggested above for the unweighted case, can all be easily extended to solve (16).

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