Mathematical Programming 18 (1980) 344–348.
North-Holland Publishing Company

SHORT COMMUNICATION

THE USE OF JACOBI'S LEMMA IN UNIMODULARITY THEORY

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Received 13 April 1978
Revised manuscript received 26 December 1979

The paper applies Jacobi's fundamental result on minors of the adjoint matrix to obtain properties on determinants of unimodular matrices.

Key words: Unimodularity, Determinants, Integral matrices, Generalized Inverses.

One of the key theorems on totally unimodular (t.u.) matrices (i.e. matrices with all minors having determinantal values equal $0, \pm 1$), is due to Gomory and reported in [3]. This result, expressed in terms of forbidden determinantal values, is used by Camion [3], to derive further characterizations of t.u. matrices, involving Eulerian matrices.

Theorem 1 (Gomory). *Let A be a* $(0, \pm 1)$*-matrix. If A is not t.u., then A has a minor whose determinant equals* ± 2.

The purpose of this note is to provide a simple proof of a generalization of Theorem 1, by applying Jacobi's fundamental lemma on minors of the adjoint. Jacobi's lemma, whose proof follows directly from the definitions, appears in most elementary books on matrices, (e.g. [1, p. 98]). Other uses of the lemma in unimodularity theory are also presented.

Lemma 1 (Jacobi). *Let G be a nonsingular matrix. Then any minor of order k in* Adj G *is equal to the complementary signed minor in* G^{T}, *multiplied by* $(\det G)^{k-1}$.

Corollary 1. *Let G be a nonsingular matrix, and let R be a proper square submatrix of* G^{-1}, *Then* $|\det R| = |\det Q|/|\det G|$, *where Q is some proper submatrix of G.*

We start by proving a generalization of Theorem 1, and then use Lemma 1, to extend other results on t.u. matrices.

Theorem 2. *Let A be an* $n \times n$, $n \geq 2$, *nonsingular real matrix satisfying*

(1) det Q *equals* $0, \pm 1$, *for all* $(n-1)\times(n-1)$ *submatrices*, Q, *of* A;
(2) det A *is integral*;
(3) $|\det Q| \geq 1$ *for some* $(n-2)\times(n-2)$ *submatrix*, Q, *of* A.
Then $|\det A|$ *equals* 1 *or* 2.

Proof. The result holds for $n=2$ if (1) is satisfied. Hence, let $n \geq 3$. First, note that Adj A is a $(0, \pm 1)$-matrix. Let D be a 2×2 (nonsingular) submatrix of Adj A, whose complementary signed minor in A^{T} is a nonsingular submatrix. Q^{T} given in (3). Then, from Lemma 1, $|\det D| = |\mathit{det}\ A|\,|\det Q|$. Since D is a 2×2 $(0, \pm 1)$-matrix the theorem follows.

Remark 1. Using the equation $|\det D| = |\det A|\,|\det Q|$ in the above proof, we conclude that if the equality holds in (3), then conditions (1) and (3) are sufficient for the result to hold. In particular, (2) is implied by (1) and (3).

Next, we illustrate by examples that none of the above conditions (1), (2) and (3) can be omitted.

The matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

shows that condition (1) cannot be dropped, while

$$A = \sqrt{2}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

indicates that (2) cannot be omitted.

To demonstrate the need for condition (3), we construct a matrix satisfying (1), (2) and whose determinant equals 4. Let H_{16} denote the Hadamard matrix of order $n=16$, (see [4]). H_{16} is a ± 1-matrix with $\det H_{16} = \pm 4^{16}$. Consider the matrix

$$B = \begin{bmatrix} H_{16} & 0 \\ 0 & e \end{bmatrix}$$

of order 17, where e is a ± 1 entry, chosen to ensure $\det B = |\det H_{16}|$. Define A by Adj $A = B$. Then $\det A = (\det B)^{1/(n-1)}$ and $A = (\det B)^{1/(n-1)} B^{-1}$, with $n = 17$. The matrix A satisfies condition (1), since Adj A is a $(0, \pm 1)$ matrix. Furthermore, $\det A = (\det B)^{1/16} = 4$.

We also comment that a matrix A satisfying (1), (2) and (3) is not necessarily integral—consider, for example,

$$A = \tfrac{1}{2}\left[\begin{array}{ccc} 1 & 2 & -1 \\ \hline -1 & 2 & 1 \\ \hline 2 & 0 & 2 \end{array}\right]$$

However, if A satisfies (1)–(3) with $|\det A| = 1$, then A is integral. This follows from the integrality of Adj A, and

$$A = \mathrm{Adj}(A^{-1})/\det(A^{-1}) = \pm\,\mathrm{Adj}(A^{-1}) = \pm\mathrm{Adj}(\mathrm{Adj}\, A/\det A).$$

Corollary 2. *Let A satisfy conditions* (1)–(3) *in Theorem* 2. *If* $|\det A| = 1\ (=2)$, *then* $|\det Q|$ *equals* 1 *or* 2 (1 *or* $\frac{1}{2}$) *for all* $(n-2)\times(n-2)$ *nonsingular submatrices Q of A.*

Proof. This follows directly from $|\det A|\,|\det Q| = |\det D|$, where D is a 2×2 nonsingular $(0,\pm 1)$-matrix.

Corollary 3. *Let A be a real* $n\times n$, $n \geq 2$, *nonsingular matrix with all its* $(n-1)\times(n-1)$ *and* $(n-2)\times(n-2)$ *minors having determinants equal* $0, \pm 1$. *If* $|\det A| \neq 1$, *then* $|\det A| = 2$ *and all the elements of* Adj A *are* ± 1.

Proof. First we use Remark 1 and the fact that for a nonsingular matrix not all $(n-2)\times(n-2)$ minors can vanish simultaneously, to argue that $|\det A| = 2$.

Let Adj $A = (b_{ij})$, and suppose that $b_{ij} = 0$ for some i and j. Since A is nonsingular, there exist indices k and m such that $b_{ik} = \pm 1$, $b_{mj} = \pm 1$. Hence, D, given by

$$\begin{bmatrix} b_{mj}, & b_{mk} \\ b_{ij}, & b_{ik} \end{bmatrix},$$

is nonsingular and $|\det D| = 1$, contradicting $|\det D| = |\det A|\,|\det Q| = 2|\det Q|$, for some $(n-2)\times(n-2)$ submatrix Q of A.

Theorem 1, motivates the following definition:

Definition 1. A matrix G, consisting of more than one element is *almost totally unimodular* (a.t.u.) if G is not t.u., but any proper submatrix of G is t.u.

From Theorem 2 or Theorem 1 we know that a.t.u. matrices, G, are square and $|\det G| = 2$. From Corollary 3 we also conclude that G^{-1} consists only of the elements $\pm\frac{1}{2}$. (This is also shown in [5].) Further properties of a.t.u. matrices are implied by the characterizations of t.u. matrices in [3]; e.g. the sum of the elements of an a.t.u. matrix is 2 (mod 4).
Using Lemma 1, we discover the following property of a.t.u. matrices.

Theorem 3. *A square matrix G is a.t.u. if and only if* G^{-1} *exists and every nonsingular submatrix, R, of* G^{-1} *has* $|\det R| = \frac{1}{2}$.

Proof. If G is a.t.u. then, G^{-1} exists and $|\det G^{-1}| = \frac{1}{2}$. Thus, it suffices to show that every proper submatrix of G has determinant $(0, \pm 1)$ if and only if every proper submatrix of G^{-1} has determinantal value of $(0, \pm\frac{1}{2})$. But the latter is implied by Corollary 1.

Finally we conclude from Corollary 1 that if G is a square nonsingular t.u. matrix, so is G^{-1}. The question is whether this property is preserved for general t.u. matrices when one considers the Moore–Penrose generalized inverse (see [2]). The answer is negative. In fact, given an $m \times n$ matrix A with no zero rows or columns we show that this inverse, A^*, is t.u. if and only if Rank $A = m = n$. The sufficiency part is explained above. The necessity will follow form the next theorem.

Theorem 4. *Let A be an $m \times n$ matrix whose entries are integer, and suppose that A contains neither zero columns nor zero rows. Then, A^*, the Moore–Penrose inverse of A, contains non integer elements if either one of the two equalities, $m = n =$ Rank A, is violated.*

Proof. With no loss of generality let $m \le n$, and let A be partitioned as $A = [A_1, A_2]$, where A_1 is $m \times r$ and Rank $A_1 =$ Rank $A = r$. Since A_1 forms a basis of A there exists a matrix D, such that $A_1 D = A_2$. Define an $m \times r$ matrix $F = A_1$, and an $r \times n$ matrix $G = [I, D]$, where I is the identity. Then Rank $F =$ Rank $G = r$, and $A = FG$. From Macduffee's theorem [2, p. 23], we obtain $A^* = G^{\mathrm{T}}(GG^{\mathrm{T}})^{-1}\,(F^{\mathrm{T}}F)^{-1}F^{\mathrm{T}}$. Let B be an $r \times r$ nonsingular submatrix of A_1, consisting of the rows $i_1, \ldots, i_r$ of A_1. Let B^* be an $r \times r$ submatrix of A^*, defined by the first r rows of A^* and the columns $i_1, \ldots, i_r$. Then

$$|\det B^*| = |\det(GG^{\mathrm{T}})^{-1}|\,|\det(F^{\mathrm{T}}F)^{-1}|\,|\det B^{\mathrm{T}}|.$$

Suppose, that either $r \le m < n$ or $r < m = n$. Then, from the Binet–Cauchy theorem [1, p. 85], in both cases A_2 is not vacuous and $\det(GG^{\mathrm{T}}) > 1$. Also, $\det(F^{\mathrm{T}}F) \ge \det(B^{\mathrm{T}}B) \ge |\det B|$, where the latter inequality is due to the integrality of the matrix B. Hence, we obtain

$$0 < |\det B^*| = \frac{|\det B|}{|\det GG^{\mathrm{T}}|\,|\det F^{\mathrm{T}}F|} < 1,$$

which in turn implies that B^* (and hence A^*) is not integer.

Remark 2. Finally, we comment that Lemma 1 can also be used to prove that if A is t.u., then every minor of A^* has determinantal value with absolute value not exceeding 1.

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