Reprinted from JOURNAL OF ECONOMIC THEORY Vol. 15, No. 2, August 1977

A Further Note On Market Equilibrium With Fixed Supply

In [3] Sandberg considered a continuous nonnegative demand function $Q(p) : E_+^n \to E_+^n$ satisfying:

$$\text{For each } i,\ i = 1,..., n \text{ and } v, u \text{ in } E_+^n$$
$$v \geqslant u \text{ and } v_i = u_i \text{ implies } Q_i(v) \geqslant Q_i(u). \tag{1}$$

Using this demand function Sandberg proposed the next theorem.

THEOREM. *There exists a unique short-run equilibrium for each s in E_+^n if and only if*

(a) *for each pair of vectors v and u such that $v \geqslant u \geqslant 0$ and $u \neq v$ we have* $Q_i(v) < Q_i(u)$ for some component i,

(b) for each $s \in E_+^n$, there exists $u \geqslant 0$ such that $Q(u) \leqslant s$.

Although we do not challenge the validity of the statement of the theorem we claim its emptiness. We will show that there is no nonnegative demand function satisfying (1) as well as (a), (b). We first demonstrate the equivalence of (a) to inverse antitonicity of $Q(u)$; i.e., for each u, v in E_+^n, $Q(v) \geqslant Q(u)$ implies $v \leqslant u$. Assuming that for some $v \geqslant u \geqslant 0$, $v \neq u$ we have $Q(v) \geqslant Q(u)$, we apply the inverse antitonicity to obtain $v \leqslant u$. Hence, inverse antitonicity implies (a). (Note that (1) is not required for this part of the equivalence.) To see the other part, suppose that for some $v \geqslant 0$, $u \geqslant 0$ satisfying $Q(v) \geqslant Q(u)$, there exists an index i such that $v_i > u_i$. Define the vector w by $w_j = \max(v_j, u_j)$, $j = 1,..., n$ then $w \geqslant v$, $w \geqslant u \geqslant 0$, and $w \neq u$. Using (1) we observe that $Q_j(w) \geqslant Q_j(v) \geqslant Q_j(u)$ if $w_j = v_j$ and $Q_j(w) \geqslant Q_j(u)$ if $w_j = u_j$. Hence $Q(w) \geqslant Q(u)$, contradicting (a).

To see the emptiness of the Theorem suppose that (1) as well as (a) and (b) are met. Then, there exists $p \geqslant 0$ such that $Q(p) \leqslant 0$. Let $u \geqslant 0$ be arbitrary, then the nonnegativity of Q yields $Q(u) \geqslant 0 \geqslant Q(p)$. Inverse antitonicity then implies the contradiction $u \leqslant p$ for all $u \geqslant 0$.

To remedy the deficiency of the Theorem we suggest the following formulation.

*THEOREM. *Let $Q(u)$: $E_+^n \to E_+^n$ be a continuous function satisfying* (1). *Condition* (a) *holds if and only if for each s in E^n nonemptiness of* $\{u \mid u \geqslant 0, Q(u) \leqslant s\}$ *implies the existence of a unique short-run equilibrium.*

Proof. Suppose that (a) holds, then, as shown above, $Q(u)$ is inverse antitone. The existence of an equilibrium follows from [1, Theorem 4.5] or [2, Theorem 3.5] or [4, Theorem 3.2]. The uniqueness is implied by [4, Corollary 3.5]. To prove sufficiency we show that Q is inverse antitone. Suppose that $Q(v) \geqslant Q(u)$ and $u \geqslant 0$, $v \geqslant 0$. Taking $s = Q(v)$ we obtain that v is the unique equilibrium. Using [4, Corollary 3.3], we have $v \leqslant u$. This completes the proof.

A final comment is in order. The unique short-run equilibrium is also the least price in the set of feasible prices; i.e., if p^* is the equilibrium for a given s then $p^* \leqslant u$ for all u in $\{u \mid u \geqslant 0, Q(u) \leqslant s\}$. The latter result follows from [4, Corollary 3.3].

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Received: May 7, 1976

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.