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ON THE SOLUTION OF DISCRETE BOTTLENECK PROBLEMS

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We consider a class of discrete bottleneck problems which includes bottleneck integral flow problems. It is demonstrated that a problem in this class, with k discrete variables, can be optimized by solving atmost k problems with real valued variables.

Several papers dealing with bottleneck flow problems [1, 3–7, 9] have recently appeared in the Operations Research literature. Most of these papers have focused on finding polynomial (in k, the number of variables) algorithms for problems with real-valued flows. A minimax problem with integral flows is presented and polynomially solved in [5]. However, the polynomial bound of the solution procedure in [5] depends on data such as arc capacities and flow weights.

In this note we present a simple observation demonstrating that optimal bottleneck integral flows can be obtained by solving O(k) problems with real valued flows. (The reader is referred to [2] for results on general combinatorial bottleneck problems.)

Consider the following bottleneck problem:

$$\begin{aligned}&\operatorname*{Min}_{x}\ \operatorname*{Max}_{1\le j\le k}\ \{d_j x_j\},\\ &Ax \ge b, \qquad\qquad\qquad\qquad\qquad\qquad (1)\\ &0\le x_j\le a_j \quad \text{and} \quad x_j \text{ integer}, \qquad j=1,\ldots,k,\end{aligned}$$

where all data are integral and $d_j \ge 0$, $j=1,\ldots,k$.

Let $J=\{j \mid d_j>0\}$. Then (1) reduces to finding the minimum z, $\bar{z}$, such that (2) has an integral solution.

$$\begin{aligned}&Ax\ge b,\\ &0\le x_j\le \min(a_j, z/d_j), \quad j\in J, \qquad\qquad (2)\\ &0\le x_j\le a_j, \qquad\qquad\qquad j\notin J.\end{aligned}$$

(Note that z/d_j can be replaced by $\lfloor z/d_j \rfloor$, while testing (2) for integral feasibility).

In the sequel we make the assumption that for all integral b and w $\{x \mid Ax \ge b,$

$0 \le x \le w\}$ is empty or else contains an integral solution. (Obviously, total unimodularity of A ensures this property.)

Lemma 1. *Let $\bar{z}$ (z^*) be the minimum z such that* (2) *has an integral* (*real-valued*) *solution. Then $\bar{z} = \bar{k}d_j$ for some integer $\bar{k}$ and $j \in J$. Also*

$$z^* \le \bar{z} \le z^* + \max_{j \in J} \{d_j\}.$$

Proof. The equality $\bar{z} = \bar{k}d_j$ is obvious from the monotonicity of $\min(a_j, z/d_j)$ in z. Let $\bar{d} = \max_{j \in J} \{d_j\}$. In order to prove that $\bar{z} \le z^* + \bar{d}$, it suffices (due to our assumption on A) to demonstrate the nonemptiness of

$$\begin{aligned} &Ax \ge b, \\ &0 \le x_j \le \min(a_j, \lfloor (z^* + \bar{d})/d_j \rfloor), \quad j \in J, \\ &0 \le x_j \le a_j, \qquad\qquad\qquad\qquad\quad j \notin J. \end{aligned} \tag{3}$$

But, from

$$\frac{z^*}{d_j} \le \left\lfloor \frac{z^*}{d_j} + 1 \right\rfloor \le \left\lfloor \frac{z^* + \bar{d}}{d_j} \right\rfloor, \quad j \in J,$$

it follows that the solution x^*, yielding z^*, is also feasible for (3).

Theorem. *Let $\bar{z}$ and z^* be as in Lemma* 1, *and $\bar{d} = \max_{j \in J} \{d_j\}$. Let $m = \lfloor z^*/\bar{d} \rfloor$. Then $\lfloor \bar{z}/\bar{d} \rfloor \in \{m, m+1\}$. Furthermore, $\lfloor \bar{z}/\bar{d} \rfloor = m$ if and only if* (2) *has an integral solution with $z = (m+1)\bar{d} - 1$.*

Proof. The inequalities $z^* \le \bar{z} \le z^* + \bar{d}$ imply that $\lfloor \bar{z}/\bar{d} \rfloor \in \{m, m+1\}$. Moreover, $\lfloor \bar{z}/\bar{d} \rfloor = m$ if and only if there exists z° in the interval $[z^*, (m+1)\bar{d})$, such that (2) has an integral solution with $z = z^\circ$. Since we seek for an integral solution, z° can be restricted to integral values. Using the monotonicity (in z) of the region defined by (2), z° can be assumed to be $(m+1)\bar{d} - 1$.

We now apply the Theorem to find $\bar{z}$. First we note that by our assumption on A, testing (2) for integral feasibility amounts to testing whether

$$X(z) = \{x \mid Ax \ge b, 0 \le x_j \le \min(a_j, \lfloor z/d_j \rfloor), j \in J; 0 \le x_j \le a_j, j \notin J\},$$

contains a real-valued solution.

A procedure for finding $\bar{z}$

Given the initial set J, go to Step 1.

Step 1. For the current set J, compute z^*, the minimum value of z for which (2) (with the current set J) has a real valued solution. Also, set $\bar{d} = \max_{j \in J} \{d_j\}$. Suppose that $\bar{d}=d_{j_0}$, set $m=\lfloor z^*/\bar{d}\rfloor$. If $X(z)$ is empty for $z=(m+1)\bar{d}-1$ go to Step 3, otherwise go to Step 2.

Step 2. If $|J|=1$, $\bar{z}=m\bar{d}$. Terminate. Otherwise, replace the upper bound a_{j_0} by $\min(a_{j_0}, m)$, and omit j_0 from the current set J. Return to Step 1.

Step 3. If $|J|=1$, $\bar{z}=(m+1)\bar{d}$. Terminate. If $X(z)$ is nonempty for $z=(m+1)\bar{d}$, set $\bar{z}=(m+1)\bar{d}$, and terminate. Otherwise, replace a_{j_0} by $\min(a_{j_0}, m+1)$, omit j_0 from the current set J. Return to Step 1.

The initial set J satisfies $|J|\le k$. Thus, the entire process terminates in atmost k iterations, since the cardinality of the current set J is reduced by 1 at each iteration. In each iteration we find z^*, the optimal value for the respective real valued problem and test the feasibility of $X(z)$ for atmost two values of z.

Using Lemma 1 we note that $\bar{z}$ can also be obtained by testing the feasibility of $X(z)$ for integers z in the range $[z^*, z^*+\bar{d}]$, $\bar{d}=\max_{j\in J}\{d_j\}$. Applying a binary search on this range, $\bar{z}$ is found after solving $O(\log\bar{d})$ feasibility problems.

Although we have addressed only minimax problems our approach is easily adapted to solve maximin problems as well.

Finally we relate our results to the bottleneck flow problems considered in [1,3–7,9]. These problems satisfy the assumption of our model since the respective matrix A is totally unimodular. (The flow model actually requires $Ax=b$ rather than $Ax\ge b$. But since $[{}^{\;A}_{-A}]$ is also totally unimodular $Ax=b$ can be replaced by $Ax\ge b$, $-Ax\ge -b$, and the general model (2) applies to the flow problem as well.) We also note that in the flow problems testing the feasibility of (2) for a given z reduces to finding a maximum flow between two nodes of the network.

Let $G=(N,E)$ be a network where N is the set of nodes and E is the set of directed arcs connecting them. Let $n=|N|$ and $e=|E|$. Given nonnegative integers $\{c(i,j)\}$, $\{w(i,j)\}$, $(i,j)\in E$, and v^* the maximum flow between nodes s and t, the weighted minimax flow problem, [5], is:

$$\text{Min}\left(\max_{(i,j)\in E} w(i,j)f(i,j)\right),$$

$$\text{s.t.}\quad \sum_{(i,j)\in E} f(i,j)-\sum_{(j,i)\in E} f(j,i)=\begin{cases}0, & i\ne s,t,\\ v^*, & i=s,\\ -v^*, & i=t,\end{cases}\tag{4}$$

$$0\le f(i,j)\le c(i,j),\qquad (i,j)\in E.$$

An $O(n^6)$ algorithm for the above problem, when flows are real-valued, is suggested in [6]. We note that the parametric approach for general combinatorial problems given in [10], also yields the same bound for this problem. In fact, the bound in [10] is $O((C(n,e))^2)$, where $C(n,e)$ is the complexity of finding a maximum flow on a graph with n nodes and e arcs. Currently $C(n,e)=O(n\min(n^2, e\log n))$, [8,11]. The case when flows are restricted to integral values is considered in [5]. There, a polynomial algorithm depending on $\{w(i,j)\}$, $\{c(i,j)\}$ is given. Our approach for solving the integral flow problem has an $O(e(C(n,e))^2)$ bound, since the real valued

bottleneck problem (Step 1 of our procedure), is solved $O(e)$ times.
The bottleneck sharing problem [1,7] is a special case of (4). Its real-valued version is obtained in $O(|T|c(n,e))$ time, where $|T|$ is the number of sinks in the network, [3,7]. Using our scheme, the case where flows must be integral will be solved in $O(|T|^2C(n,e))$ effort.

We conclude by conjecturing that our scheme can be applied internally by a (parametric) algorithm for the real valued bottleneck poblem to yield the same time bound for the integral-valued version as well.

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