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ALGEBRAIC OPTIMIZATION: THE FERMAT-WEBER LOCATION PROBLEM

R. CHANDRASEKARAN
University of Texas, Dallas, TX, USA

A. TAMIR
New York University, NY, USA, and Tel Aviv University, Tel Aviv, Israel

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The Fermat-Weber location problem is to find a point in $\mathbb{R}^n$ that minimizes the sum of the (weighted) Euclidean distances from m given points in $\mathbb{R}^n$. In this work we discuss some relevant complexity and algorithmic issues. First, using Tarski's theory on solvability over real closed fields we argue that there is an infinite scheme to solve the problem, where the rate of convergence is equal to the rate of the best method to locate a real algebraic root of a one-dimensional polynomial. Secondly, we exhibit an explicit solution to the strong separation problem associated with the Fermat-Weber model. This separation result shows that an ε-approximation solution can be constructed in polynomial time using the standard Ellipsoid Method.

Key words: Algebraic optimization, location theory, ellipsoid algorithms.

Let $a^1, \ldots, a^m$ be m distinct points in $\mathbb{R}^n$. Suppose that each point a^i, $1 \leq i \leq m$, is associated with a positive weight w_i. The Fermat-Weber location problem [11, 17] is to find a point in $\mathbb{R}^n$ that will minimize the sum of the (weighted) Euclidean distances from the m given points:

$$\operatorname{Min} f(x) = \sum_{i=1}^{m} w_i \|x - a^i\|. \tag{1}$$

It is well known that if the data points are not collinear the objective is strictly convex, and therefore has a unique optimum. (In the collinear case at least one of the points $a^1, \ldots, a^m$ is optimal and it can be found in linear time by the algorithm in [3].)

Suppose that the input is rational and that the points $a^1, \ldots, a^m$ are not collinear. (1) can be formulated as the following algebraic program in $\mathbb{R}^{n+m}$:

$$\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{m} w_i s_i \\
&\text{subject to} && \sum_{j=1}^{n} (x_j - a_j^i)^2 \leq s_i^2, \quad i = 1, \ldots, m, \\
& && s_i \geq 0.
\end{aligned} \tag{2}$$

From the general theory developed by Tarski [16] on solvability over real closed fields, it follows that (2) has an optimal solution which is algebraic. Let f^* and x^* denote the optimal objective value and an optimal solution to (2) respectively. f^* and each component of x^* can be associated with their respective characteristic polynomials. (Recall that if t is algebraic its characteristic polynomial is the polynomial of minimum degree which has t as one of its roots, and all its coefficients are integer with 1 as their greatest common divisor. The degree of t is the degree of its characteristic polynomial and the height of t is the largest of the absolute values of the coefficients of this polynomial.)

It follows from Tarski [16] and the more recent algorithms of [2, 4, 10] that one can obtain the characteristic polynomials of f^*, and the components of x^*, using only a finite number of elementary rational operations, i.e., $+$, $-$, $\times$, $/$, and comparisons between rational numbers. Furthermore, each such polynomial will be associated with an interval where the respective algebraic number, f^* or x^*, is its only root. Thus, these polynomials with their respective intervals are used to characterize and represent the solution to (2).

In general the algorithms in [2, 4, 10, 16] are not efficient. In the worst case they are at least exponential. However, in our opinion, the finiteness of these procedures questions the appropriateness of using concepts like rate of convergence and average rate [14] to measure the effectiveness of solution procedures. These terms measure the local rapidity of the convergence of an infinite sequence of iterates to the optimal solution. They refer only to the tail of the sequence. Also, the convergence rate concepts do not take into consideration the computational effort per iteration as well as the sizes of the iterates, i.e., their lengths in binary encoding. Indeed, suppose that one is allowed to ignore both aspects, i.e., a finite initial computational effort, and the sizes. Using Tarski's approach we can claim that there is an infinite scheme to solve (2), where the rate of convergence is equal to the rate of the best method to locate a real algebraic root of a one-dimensional polynomial, and one cannot hope to do better because of the nature of the solution. First find the characteristic polynomials of f^* and x^*, with their respective intervals. Secondly, apply an efficient (infinite) scheme (e.g., Newton's) to locate the unique root of each polynomial in its respective interval.

Finding the degree and the height of the solution to (2) has become a challenging problem since it is related to classical problems in algebraic number theory [1, 5, 6]. The best bound known on the degree is still the exponential (in m) bound which can be obtained by elementary methods. There are a couple of theoretical results on the algebraic nature of the solution. Melzak [13] proved that even for the unweighted case, $w_i = 1$ for all i, this solution cannot in general be obtained by a straight edge and compass when $n = 2$ and $m = 5$. Recently Bajaj [1] extended this result and proved that the solution point cannot be expressed in radicals when $n = 2$ and $m = 5$.

We have already mentioned above that when the algebraic characterization of the optimal solution is available ε-approximation solutions can be obtained fficiently by standard and traditional root finding procedures. Thus, it is most esirable to find this characterization. However, in view of the above complexity esults, the task of characterizing f^*, the optimal objective value to (2), seems to e very difficult.

Suppose that k and M denote some upper bounds on the degree and height of * respectively. Let L be the length in binary encoding of the input to (2). From ne results in [9] we conclude the following. The characteristic polynomial of f^* an be obtained in time which is polynomial in k, $\log M$ and L, provided that for ny given rational $\varepsilon > 0$ it takes polynomial time (in L and the length of ε) to find point x in $\mathbb{R}^n$ with $f(x) \leq f^* + \varepsilon$. Indeed, we will demonstrate that ε-approximation olutions can be constructed by polynomial schemes. Unfortunately, as stated above, he best known bounds on k and M are still exponential in m. (Note, however, that hen m is fixed the characteristic polynomial can be constructed in polynomial ime in L.)

The existence of polynomial ε-approximation algorithms follows, for example, rom [12, Theorem 2.2.15]. We can easily verify that the oracles there can be made olynomial when applied to the Fermat-Weber problem. In particular, since the ptimal solution is in the convex hull of the points $a^1, \ldots, a^m$, we can restrict the earch to some well defined box $A \leq x \leq B$ which contains the convex hull. (A and B are easily computable integer vectors with $A < B$).

The polynomial scheme in [12] uses the Shallow Cut Ellipsoid Method which is ased on "very weak separation". This very weak separation condition was not ufficient to ensure the applicability of the simpler and standard Ellipsoid Method as described in [7]). We will next show that for the Fermat-Weber problem, as ormulated by (2), there is an explicit form solution for the (ε-independent) strong eparation problem. With that result we can then use the standard Ellipsoid Method 7, Theorem 3.1] to obtain an ε-approximation in polynomial time. (Because of its elative simplicity it seems that the standard method is more efficient than the hallow Cut Ellipsoid Method.)

We now assume that the x variables in (2) are confined to the box $A \leq x \leq B$, which contains the convex hull of the points $a^1, \ldots, a^m$. (A and B are explicitly nown integer vectors with $A < B$.) It then follows that the variables $s = (s_1, \ldots, s_m)$ an be restricted to the box $0 \leq s_i \leq \|B - A\| + 2 \equiv C$, $i = 1, \ldots, m$.

Reformulate the location model (2) as:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{m} w_i s_i \\
\text{subject to} \quad & \|x - a^i\| \leq s_i, \quad i = 1, \ldots, m, \\
& A \leq x \leq B, \\
& 0 \leq s_i \leq C, \quad i = 1, \ldots, m.
\end{aligned}
\tag{3}
$$

Let $X \subseteq \mathbb{R}^{n+m}$ denote the feasible set in (3). To solve the strong separation problem with respect to X it will suffice to consider the separation with respect to the following cone S_2 in $\mathbb{R}^{n+1}$,

$$S_2 = \{(x, t) \mid x \in \mathbb{R}^n, \ t \geq \|x\|\}. \tag{4}$$

Lemma 1. *Let $p \geq 1$. Consider the cone*

$$S_p = \left\{(x, t) \mid x \in \mathbb{R}^n, \ t \geq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}\right\}.$$

Suppose that (x^0, t^0), $t^0 \geq 0$, is not in S_p. Then,

$$t \sum_{j=1}^{n} |x_j^0|^p > t^0 \sum_{j=1}^{n} |x_j^0|^{p-1} (\operatorname{sign} x_j^0) x_j \tag{5}$$

for any (x, t) in S_p with $t > 0$.

Proof. Clearly $x^0 \neq 0$. Therefore, (5) is certainly satisfied when its right-hand side is nonpositive. Thus, suppose that the right-hand side of (5) is positive.

Let (x, t) be in S_p. Then we have

$$\begin{aligned} t \sum_{j=1}^{n} |x_j^0|^p &\geq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j^0|^p\right) \\ &\geq \left(\sum_{j=1}^{n} |x_j^0|^p\right)^{1/p} \sum_{j=1}^{n} |x_j^0|^{p-1} (\operatorname{sign} x_j^0) x_j \\ &> t^0 \sum_{j=1}^{n} |x_j^0|^{p-1} (\operatorname{sign} x_j^0) x_j, \end{aligned}$$

where the first inequality is obtained from the fact that (x, t) is in S_p, the second inequality is implied by Holder's inequality [8], and the strict inequality follows from the positivity of the right-hand side of (5) and the supposition that (x^0, t^0) is not in S_p. This completes the proof. □

Replacing the strict inequality in (5) by equality we obtain an hyperplane H in $\mathbb{R}^{n+1}$ which contains (x^0, t^0) and supports S_p at the origin only. To separate the origin as well we have the following.

Theorem 2. *Consider the cone S_2 in (4). Suppose that (x^0, t^0), $t^0 \geq 0$, is not in S_2.*

(i) *If $t^0 = 0$, the hyperplane $H_1 = \{(x, t) \mid \operatorname{Max}(1, \|x^0\|^2) \ t = x^0(x - x^0)\}$ strongly separates (x^0, t^0) from S_2.*

(ii) *If $t^0 > 0$, the hyperplane $H_2 = \{(x, t) \mid t(\|x^0\|^2 + (t^0)^2) + t^0(\|x^0\|^2 - (t^0)^2) = 2t^0 x^0 x\}$ strongly separates (x^0, t^0) from S_2.*

Proof. Consider the point $(x^0, \|x^0\|)$ in S_2. Since $x^0 \neq 0$, a supporting hyperplane of S_2 at $(x^0, \|x^0\|)$ is given by the equation

$$t = \|x^0\| + x^0(x - x^0)/\|x^0\|. \tag{6}$$

Let (x, t) be in S_2. Then

$$t \geq \|x^0\| + x^0(x - x^0)/\|x^0\| > t^0 + x^0(x - x^0)/\|x^0\|.$$

Therefore, the hyperplane given by (7) strongly separates (x^0, t^0) from S_2:

$$\|x^0\| t = \|x^0\| t^0 + x^0(x - x^0). \tag{7}$$

To obtain a strongly separating hyperplane with rational (with respect to (x^0, t^0)) coefficients we rotate (7) as follows. (Recall that Ellipsoid algorithms require rational data.)

(i) Suppose that $t^0 = 0$. Consider a point (x, t) in S_2. Then

$$\mathrm{Max}(1, \|x^0\|^2) t \geq \|x^0\| t > \|x^0\| t^0 + x^0(x - x^0) = x^0(x - x^0),$$

and the hyperplane $H_1 = \{(x, t) \mid \mathrm{Max}(1, \|x^0\|^2) t = x^0(x - x^0)\}$ strongly separates S_2 from (x^0, t^0).

(ii) Suppose that $t^0 > 0$. Any hyperplane which is a strict convex combination of the hyperplanes given by (7) and (5) (for $p = 2$), strongly separates S_2 from (x^0, t^0). Specifically, for each $0 < \lambda < 1$ the following hyperplane separates S_2 from (x^0, t^0):

$$\frac{t(\lambda(\|x^0\|^2 - \|x^0\| t^0) + \|x^0\| t^0)}{t^0} + (1 - \lambda)(\|x^0\|^2 - \|x^0\| t^0) = x^0 x. \tag{8}$$

From $(\|x^0\| - t^0)^2 > 0$ we obtain $\|x^0\|^2 - \|x^0\| t^0 > \frac{1}{2}(\|x^0\|^2 - (t^0)^2) > 0$. Defining λ by $(1 - \lambda)(\|x^0\|^2 - \|x^0\| t^0) = \frac{1}{2}(\|x^0\|^2 - (t^0)^2)$, and substituting in (8) yield the following rational equation for a strongly separating hyperplane:

$$\frac{t(\|x^0\|^2 + (t^0)^2)}{(2t^0)} + \frac{(\|x^0\|^2 - (t^0)^2)}{2} = x^0 x. \tag{9}$$

This completes the proof. $\square$

With the strong separation result of Theorem 2 one can directly apply the standard Ellipsoid Method [7] to generate an ε-approximation solution to (1). This approach can be extended to some generalizations of (1) involving Euclidean distances. For example, the separation results can be modified to solve the model in [15] with the additional restriction that the point x is restricted to some polyhedral set. Also, we can obtain ε-approximations to the point that minimizes the sum of weighted distances from m given polyhedral sets or ellipsoids.

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