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CONTEXTUAL AREAS

Subadditive and Homogeneous of Degree One Games Are Totally Balanced

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A cooperative game with transferable utility is said to be *homogeneous of degree one* if for any integer m, the value of cloning m times all players at any given coalition, leads to m times the value of the original coalition. We show that this property coupled with subadditivity, guarantees the nonemptyness of the core of the game and of all its subgames, namely, the game is totally balanced. Examples for games stemming from the areas of retailing and of facility location are given.

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1. Introduction

A cooperative game with transferrable utility is defined by a set of n players, $N = \{1, 2, \ldots, n\}$, that can form coalitions. Specifically, any subset S of players in N, can cooperate, where $\varnothing \subseteq S \subseteq N$ is called a *coalition*, and $S = N$ is also called the *grand coalition*. Each coalition S is associated with a real value denoted by $V(S)$, where $V(\varnothing) = 0$, that represents the total cost inflicted on the members of coalition S if they cooperate. Note that once a coalition is formed, it is irrelevant to its members what the other players are doing, and in particular, which coalitions they form. The function $V\colon 2^N \to \Re$ is called the *characteristic function* of the game. The pair $G = (N, V)$ is said to be a *cooperative game with transferable utility*. A game $G = (N, V)$ is called *subadditive* if for any two disjoint coalitions S and T, $V(S \cup T) \leqslant V(S) + V(T)$. Subadditivity ensures that the socially best partition of the players of N to disjoint coalitions is when all players cooperate and form the grand-coalition N. Subadditive games bear the concept of *economies of scope*, i.e., when each player, or set of players, contributes its own skills and resources, the total cost is no greater than the sum of the costs of the individual parts.

A prerequisite for the stability of the grand coalition is an agreement on fair allocation of the cost $V(N)$ among the players of N. Several fairness concepts have been proposed in the literature. One of the most appealing among them is the *core*: a vector $x \in \Re^n$ is said to be *efficient* if $\sum_{i=1}^n x_i = V(N)$, and it is said to be a *core cost allocation* if it is efficient and $\sum_{i\in S} x_i \leqslant V(S)$ for any $\varnothing \subseteq S \subseteq N$. The collection of all core allocations, called the *core* of the game, is a simplex that is defined by n decision variables and by $2^n - 1$ constraints. Thus, finding a core allocation for a given game, except for specific ones having special structures, may be an intricate task. Indeed, this issue coupled with the possibility that the core is empty, makes the problem of finding a core allocation, or showing that the core is empty, a real challenge.

A game whose core is nonempty is said to be *balanced*. A balanced game for which all its 2^{n-1} subgames are also balanced is said to be *totally balanced*. There exist examples that show that subadditivity by itself does not guarantee balancedness of the game. Moreover, a game that is not subadditive is also not totally balanced as for any disjoint S and T of N for which, $V(S) + V(T) < V(S \cup T)$, the subgame $(S \cup T, V)$ has an empty core since any efficient allocation of $V(S \cup T)$ among the players of $S \cup T$ will be objected by at least one of the coalitions, S or T.

We focus on sufficient conditions for a subadditive game to be totally balanced. To the best of our knowledge, the following are the only two well-known general sufficient conditions for a game to be totally balanced. In this paper, we propose a new sufficient condition for a cooperative game to be totally balanced. First, we present the two known sufficient conditions:

CONDITION 1. *Concave games*. A game $G = (N, V)$ is said to be concave if its characteristic function is concave, meaning that for any two coalitions $S, T \subseteq N$, $V(S \cup T) + V(S \cap T) \leqslant V(S) + V(T)$.

CONDITION 2. *Market games*. (See Shapley and Shubik 1969 and Chap. 13 in Osborne and Rubinstein 1994.) Suppose there are l types of inputs. An *input vector* is a nonnegative vector in $\Re^l_+$. Each of the n players possesses an initial commitment vector $w_i \in \Re^l_+$, $1 \leqslant i \leqslant n$, which states a nonnegative quantity for each input. Moreover, each player is associated with a continuous and convex cost function f_i: $\Re_+^l \to \Re_+$, $1 \leqslant i \leqslant n$. A profile $(z_i)_{i\in N}$ of input vectors for which $\sum_{i\in N} z_i = \sum_{i\in N} w_i$ is an *allocation*. Then, for any $\varnothing \subseteq S \subseteq N$,

$$V(S) = \min\left\{\sum_{i\in S} f_i(z_i):\ z_i \in \Re_+^l,\ i\in S \text{ and } \sum_{i\in S} z_i = \sum_{i\in S} w_i\right\}. \quad (1)$$

Actually, a necessary condition for a game to be totally balanced is that the game can be reformulated as a market game; see Peleg and Sudholter (2007). Therefore, in theory, the class of totally balanced games coincides with the class of market games. The snag is that if a game is not presented naturally as a market game, reformulating it as such (or showing that such a reduction is impossible), is an intricate job by itself, and hence this approach is not very useful. Thus, in the sequel, we refer to market games as games that are originally formulated as in Condition 2, or games that a reduction to the form of a market game is known.

Conditions 1 and 2 allow the characteristic function to be quite general as it may assign values to coalitions arbitrarily and independently of the profile of its members. In particular, the characteristic function may assign different values to two players that are identical in all aspects except their identity. In practice, many cooperative games are symmetric in the sense that they do not exploit this freedom of the definition, and de facto, identical players/coalitions that differ only in the identity of their members affect the characteristic function in exactly the same way. In such games it is easy to generalize the definition of the characteristic function to any set of players, not necessarily subsets of N. In a previous paper (Anily and Haviv 2012) we formalize this idea by introducing the class of *regular games* where players, except for their own name, are fully characterized by some quantitative properties. For example, a tax payer is identified by his or her social security number (identity) and is characterized by a number of quantitative properties, e.g., his or her income level, tax allowance, tax credits, etc. A game is said to be regular if the characteristic function assigns to any collection of players, not necessarily players of N, a value that is computed via a closed-form expression of the quantitative properties of its players and is not a function of their identity. In the next section we formalize this notion rigorously. In Anily and Haviv (2012) we also identify a class of regular games called *regular market games* and a reduction scheme that reduces these games to market games, proving, according to Peleg and Sudholter (2007), that they are totally balanced.

Condition 3. *Regular market games*. Follow the definition of market games in Condition 2, with the following differences: (i) The functions f_i in (1) are identical, i.e., $f_i \equiv f$ for all $i \in N$, so that the cost associated with a player is independent on its identity. (ii) The total of the commitment vectors of the players should be allocated among the players and an external agent, where the cost to the coalition due to the allocating commitments to the external agent is linear. That means that there exists a linear function h: $\Re_+^l \to \Re$, with $h(\vec{0}) = 0$, so that Equation (1) is replaced by

$$V(S) = \min\left\{\sum_{i\in S} f(z_i) + h\left(\sum_{i\in S}(w_i - z_i)\right):\ z_i \in \Re_+^l,\ i\in S \text{ and } \sum_{i\in S} z_i \leqslant \sum_{i\in S} w_i\right\}. \quad (2)$$

In addition to the above mentioned three classes of games, a few structural games have been identified in the literature as totally balanced. Of particular interest here is the class of *permutation games* (see Tijs et al. 1984 and Peleg and Sudholter 2007 subsection 3.4.2): Let $\Pi(N)$ be the set of all permutations of N. A permutation $\pi \in \Pi(N)$ is a one-to-one function from N to N, and $\Pi(S) = \{\pi \in \Pi(N):\ \pi(i) = i \text{ for } i \in N\backslash S\}$.

Condition 4. A *Permutation game* is defined by a cost function p: $N \times N \to \Re$, and a characteristic function of the form $V(S) = \min_{\pi\in\Pi(S)} \sum_{i\in S} p(i, \pi(i))$ for any $S \subseteq N$.

To present our new sufficient condition for total balancedness, we generalize a well-known property of real functions to characteristic functions of regular games, namely, the property of *homogeneity of degree p*:

Definition 1. A game is said to be homogeneous of degree p if for any integer m, the characteristic function value of cloning m times a collection of players is m^p times the value of the original collection of players.

The main result of this paper is that subadditivity and homogeneity of degree one of the characteristic function of a regular game imply total balancedness. That means, that within the class of subadditive games, five subclasses of games that are totally balanced are currently known: (i) concave games, (ii) market games, and (iii) permutation games; and within the class of regular games also (iv) regular market games and (v) homogeneous of degree one games. These subclasses are not disjoint. For example, the class of homogeneous degree one games contain the subclass of games that are both regular and permutation games. We refer to such games as *regular permutation games*. In fact, the state-of-the-art level of characterization of the core of a game depends on the class that the game belongs to: a full characterization of the core is known only for concave games; see Shapley (1971). For market games, and as a consequence also for regular market games, a single core allocation based on *competitive equilibrium prices* is proposed in chapter 13 in Osborne and Rubinstein (1994). A single core allocation for a permutation game is obtainable by solving a linear programming problem consisting of $2n$ variables and n^2 constraints; see Peleg and Sudholter (2007). Our result on total balancedness of homogeneous of degree one games, leaves open the challenge of identifying core cost allocations for such games.

The contribution of this paper stems from the fact that there exist subadditive and homogeneous of degree one games that are not concave, they have neither the form of regular market games or permutation games. This paper proves that such games are totally balanced.

In the next section we define regular games. In §3 we present two examples of regular games, one in retailing and another in facility location. The examples are of subadditive regular games that are neither concave nor are presented as regular market games or as regular permutation games, and therefore it is impossible to invoke any of the sufficient conditions mentioned above in order to prove that these games are totally balanced. In §4 we present formally the homogeneity of degree one property, and the key theorem of this paper, which states that subadditivity and homogeneous of degree one of the characteristic function of a regular game, is sufficient for total balancedness. The examples in §3 can easily be shown to be subadditive and homogeneous of degree one, proving that they are also totally balanced. In §5, the theorem is proved.

2. Regular Games

A cooperative game $G=(N,V)$ is defined by its set of players N and its characteristic function V. This definition is useful as long as one considers just the players in N. However, in general, one may want to consider adding new potential players to the game. As currently presented, the characteristic function is not applicable for such cases. In many settings this limitation is artificial as each potential player of the game is characterized by a vector of quantitative properties, hereby called a *vector of properties*, and it affects the cost of any coalition that it joins only by this vector through a closed-form mathematical expression that is independent of the player's identity.

A regular game $G=(N,V)$ (see Anily and Haviv (2012)) satisfies the following conditions: there exists $\kappa \geqslant 1$ resources, indexed by $l=1,\ldots,\kappa$. Each player $i\in N$ is fully characterized by the resources' quantities that he or she owns. That means that player $i\in N$ is associated with a vector of properties $y^i \in \Re^\kappa$, so that y^i_l denotes the quantity of resource l, $1\leqslant l\leqslant \kappa$, that he or she owns. The vector of properties of the players may be required to satisfy some feasibility constraints of the form $y\in D$, where $D\subseteq \Re^\kappa$. Some of the resources are sharable among the members of a coalition, whereas the others are nonsharable. In some games all resources are sharable. Nonsharable resources serve as attributes (parameters) of the players. The characteristic function value $V(S)$ of coalition $S\subseteq N$, is a function of the $|S|$ vectors of properties of the members of S and is otherwise independent of the identity of its members. $V(S)$ denotes the cost induced by the members of S when they share the sharable resources according to the rules of the game. Let $y^{(m)}$ denote a sequence of m vectors of properties $y^1,\ldots,y^m$ in D. The following two definitions formally define a regular game:

Definition 2. An infinite sequence of symmetric functions $V_0, V_1,\ldots,V_m,\ldots$ is said to be *infinite increasing input-size symmetric sequence* (*IIISSS*) *of functions* for given integer $\kappa\geqslant 1$, and a subset D of $\Re^\kappa$, if

- $V_0\equiv 0$;
- for any $m\geqslant 1$, $V_m\colon D^m\to\Re$;
- there exists a vector $y^0\in D$ such that $V_1(y^0)=0$ and for any given sequence of $m-1$ vectors of properties $y^{(m-1)}=(y^1,\ldots,y^{m-1})\in D^{m-1}$, $V_{m-1}(y^{(m-1)})=V_m(y^{(m-1)},y^0)$.

For a given IIISSS of functions $(V_m)_{m\geqslant 0}$, V_m receives as input m vectors of size κ, each is a member of the set D, and it returns a real value. As the functions V_m are symmetric, the order of the m input vectors has no affect on the value of the function. The third item of the definition guarantees that the definition of the various functions of the IIISSS of functions is consistent, i.e., it excludes the possibility that there exist two functions V_l and V_k for $l\neq k$, $l,k\geqslant 1$, where each is defined by a different mathematical expression. This is achieved by requiring to have a *null vector of properties* $y^0\in D$ that links the different functions through a forward recursion. For example, suppose that each player i is associated with a certain score α_i and the value of a coalition is the average score of its members. In such a case let $\kappa=2$, player i is associated with a vector $y^i=(\alpha_i,1)$, the null vector is $y^0=(0,0)$ and $D=\{(0,0)\}\cup\{(x,1)\colon x\in\Re\}$. Given m vectors of properties $y^{(m)}\in D^m$, $y^i=(\alpha_i,\beta_i)\in y^{(m)}$, $i=1,\ldots,m$, the value $V_m(y^{(m)})=\sum_{i=1}^m \alpha_i/\sum_{i=1}^m\beta_i$, i.e., $V_m(y^{(m)})$ is the average score of the nonnull vectors in D. Note that the choice of y^0 as the zero vector is a quite natural choice for a null vector that holds in many other games. But in some games y^0 is not necessarily the zero vector. Consider a similar example to the above one with a characteristic function that for any coalition returns the product of the scores in the coalition divided by the number of players in the coalition, i.e., $V_m(y^{(m)})=\Pi_{i=1}^m\alpha_i/\sum_{i=1}^m\beta_i$. In such a case the null vector y^0 is $(1,0)$, and $V_1(y^0)$ is defined as 0.

Definition 3. A game $G=(N,V)$ is called regular if there exists a set $D\in\Re^\kappa$, such that player i, $i\in N$, is associated with a vector of properties $y^i\in D$, and there exists an IIISSS of functions $V_l\colon D^l\to\Re$, $l\geqslant 0$, such that for any $S\subseteq N$, $V(S)=V_{|S|}(y^i|_{i\in S})$.

Observation 1. A market game $G=(N,V)$, as described in §1, is not a regular game in general, as the cost function of a player may depend on his or her identity. A market game is a regular game if all individual cost functions $f_i(\cdot)$, $i\in N$, are identical, i.e., $f_i\equiv f$ for all $i\in N$.

The subadditivity of an IIISSS of functions is defined as follows:

Definition 4. An IIISSS of functions $V_0,V_1,V_2,\ldots$ is said to be *subadditive* if for any two finite sequences of vectors of properties in D, $(y^i_A)|_{i\in A}$ and $(y^i_B)|_{i\in B}$, $V_{|A|+|B|}((y^i_A)|_{i\in A},(y^i_B)|_{i\in B})\leqslant V_{|A|}((y^i_A)|_{i\in A})+V_{|B|}((y^i_B)|_{i\in B})$.

In the next section we consider two examples of regular cooperative games that do not fit the structure of these two types. In addition, none of the them can be proved to be totally balanced by using the sufficient conditions described in §1.

3. Examples

The first example deals with a situation that we often encounter in sales, where we do not have to pay for all the items that we buy. Suppose that the items in the store are partitioned into k categories for some $k \geqslant 2$. If a customer buys k items, one from each category, then she gets for free one of the items whose price is the cheapest among the k items she picked up. We call such a sale a $(k-1)+1$ sale.

Example 1. Consider a department store that announces a $(k-1)+1$ sale for k categories of items. The sale is such that if one buys one item from each category, she gets for free one item whose price is the cheapest. Suppose the categories are indexed by $1, \ldots, k$. A customer that takes advantage of the sale is associated with a vector of properties of size k, $(x_1^i, \ldots, x_k^i) \in \Re_+^k$, where x_j^i is the price of the item in category j that customer i picks up. This customer, if acting individually, pays $\sum_{j=1}^k x_j^i - \min\{x_j^i\colon j = 1, \ldots, k\}$. Cooperation among customers of a group $S \subseteq N$ may generate some savings. Let $V(S)$ denote the minimum payment that coalition S can achieve by reassigning the items of the different categories among its members.

To present the game as a regular game let the set of feasible vectors of properties be $D = \{(x_1, \ldots, x_k)\colon x_j \geqslant 0 \; j = 1, \ldots, k\}$, where the null vector is the zero vector. Let $\Pi(N)$ be the set of all permutations of N, and $\Pi(S) = \{\pi \in \Pi(N)\colon \pi(i) = i \text{ for } i \in N \backslash S\}$. Let also $\sigma_1, \ldots, \sigma_k$ be k permutations in $\Pi(N)$, where permutation σ_1 is the identity permutation, i.e., $\sigma_1(i) = i$ for $i = 1, \ldots, n$. The characteristic function of the $(k-1)+1$ sale game is defined by

$$V(S) = \sum_{i \in S} \left[\sum_{j=1}^k x_j^i - \max_{\sigma_l \in \Pi(S),\, l=2,\ldots,k} \min\{x_1^{\sigma_1(i)}, x_2^{\sigma_2(i)}, \ldots, x_k^{\sigma_k(i)}\} \right] \qquad \varnothing \subseteq S \subseteq N.$$

For any integer $m \geqslant 1$, let $\mathcal{P}(m)$ be the collection of all permutations of the sequence $(1, \ldots, m)$, where permutation σ_1 is the identity permutation. Define the IIISSS of functions V_m, $m \geqslant 0$, as follows: $V_0 = 0$, $V_1(x_1, \ldots, x_k) = \sum_{j=1}^k x_j - \min\{x_j\colon j = 1, \ldots, k\}$, and for $m \geqslant 1$

$$V_m((x_1^i, \ldots, x_k^i)_{i=1\ldots m}) = \sum_{i=1}^m \left[\sum_{j=1}^k x_j^i - \max_{\{\sigma_l \in \mathcal{P}(m),\, l=2,\ldots,k\}} \min\{x_j^{\sigma_j(i)}\colon 1 \leqslant j \leqslant k\} \right]. \quad (3)$$

We provide now a simple procedure that determines the characteristic function value for any coalition and for $k > 1$ categories of items. Claim 1 specifies an optimal sequence of permutations that minimize (3):

Claim 1. *Let $\sigma_j \in \mathcal{P}$ for $j = 1, \ldots, k$ be k permutations, with σ_1 being the identity permutation, and $x_j^{\sigma_j(1)} \leqslant \cdots \leqslant x_j^{\sigma_j(m)}$ for $j = 1, \ldots, k$. Then, (3) is equivalent to*

$$V_m((x_1^i, \ldots, x_k^i)_{i=1}^m) = \sum_{i=1}^m \sum_{j=1}^k x_j^i - \sum_{i=1}^m \min\{x_j^{\sigma_j(i)}\colon 1 \leqslant j \leqslant k\}. \quad (4)$$

Proof. The proof is by induction on m, the size of the coalition. For $m = 1$ the proof is trivial. Consider $i = m$ and the following vector of properties $\bar{x} = (x_1^m, x_2^{\sigma_2(m)}, \ldots, x_k^{\sigma_k(m)})$ that consists of the prices of the most expensive item in each of the k categories. In any reassignment of the items among the customers of the coalition it is necessary that the coalition pays for all the $k-1$ most expensive items in $\bar{x}$. Thus, without loss of generality, we assign these $k-1$ items to customer m. Hence, customer m is assigned items in all categories except for some category $l \in \{1, \ldots, k\}$. Yet, it is easy to see that it is optimal to assign to her also the most expensive item of category l, now without a charge. In other words, the vector $\bar{x}$ is assigned completely to customer m. The problem then repeats itself with a coalition of $m-1$ customers, and the remaining set of items in each of the k categories. The proof then follows by induction.

The IIISSS of functions given in (3) is subadditive as when an optimization problem is involved in defining the functions $(V_m)_{m \geqslant 1}$ then the optimal solution for a set A of vectors of properties coupled with the optimal solution for a disjoint set B of vectors of properties is still feasible for $A \cup B$. Yet, a better solution can be found for $A \cup B$.

The following instance of a $1+1$ sale game shows that $(k-1)+1$ sale games are not concave; see Condition 1 in §1. Let $N = \{1, 2, 3\}$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 10$, $\beta_1 = \beta_2 = 10$, and $\beta_3 = 1$. Let $S = \{1, 3\}$ and $T = \{2, 3\}$. Thus, $V(S) = V(T) = 11$, $V(S \cap T) = V(\{3\}) = 10$, and $V(S \cup T) = 1 + 10 + 10 = 21$, implying that $V(S \cup T) + V(S \cap T) > V(S) + V(T)$.

The formulation of the $(k-1)+1$ sale game in (3) does not look as a formulation of a market game (see Condition 2 in §1) or as a formulation of a regular market games (see Condition 3 in §1) as we deal here with a discrete rather than a continuous optimization. The special case of a $1+1$ sale game with two categories, and one pays for the most expensive item of the two, is a regular permutation game and therefore it is totally balanced; see §1. In the next section we show that also the general $(k-1)+1$ sale game is totally balanced.

Our second example is from the area of location of service facilities:

Example 2. Suppose that a number of towns that are part of a bigger metropolitan need the service of a fire station. Each can have its own station at a dedicated location. Cooperation can take place when a number of towns use the same station. The goal is to minimize the sum of the distances between the centers of the towns and their closer stations. Specifically, let $x_i \in \Re^2$ be the center of town i and let $y_i \in \Re^2$ be the default location of its fire station. Define $V_0 = 0$ and

$$V_m((x_i, y_i), 1 \leqslant i \leqslant m) = \sum_{i=1}^m \min_{j \mid 1 \leqslant j \leqslant m} ||x_i - y_j||^2. \quad (5)$$

In particular $V_1(x, x) = 0$ for any $x \in \Re^2 \times \Re^2$. The sequence V_m for $m \geqslant 0$ is an IIISSS of functions with $D = \{(x, y) \in \Re^2 \times \Re^2\colon \max\{\|x\|, \|y\|\} \leqslant \rho\} \cup \{(z, z) \in \Re^2 \times \Re^2\colon \|z\| > 2\rho\}$ for some constant $\rho > 0$. The pair (z, z) serves as the null vector and therefore it is chosen so that $\|z\|$ is a large number relative to the input implying that pairing a town in D with (z, z) is never optimal.

In view of the above presentation the game is regular. It is easy to see that this game is subadditive. The following example shows that it is not a concave game. Let $N = \{1, 2, 3\}$ and assume $x_1 = x_2 = x_3 = (1, 0)$, that $y_1 = y_2 = (1, 0)$ and that $y_3 = (10, 0)$. Let $S = \{1, 3\}$ and $T = \{2, 3\}$. Thus, $V((1, 3)) = V((2, 3)) = 0$, $V(S \cup T) = 0$ and $V(S \cap T) = 81$. Clearly, $V(S) + V(T) < V(S \cap T) + V(S \cup T)$, refuting a possible conjecture that this game is concave. For the same reasons as Examples 1, this game does not look as neither a market game nor as a regular market game, or a permutation game.

The new condition that we present in §4 will easily prove the total balancedness of these three cooperative regular games.

4. Homogeneity of Degree One

We now present the homogeneity of degree one property for regular games. For this sake we need the following notation:

Definition 5. Given a regular game $G = (N, V)$ that is associated with $D \subseteq \Re^{\kappa}$, and a sequence A of vectors of properties in D let $A^{(p)}$ be a set of vectors of properties in D containing p replicas of any member in A.

Definition 6. An IIISSS of functions $(V_m)_{m \geqslant 0}$ (with the corresponding set D) is said to be *homogeneous of degree p*, $p \geqslant 0$, if for any given sequence A of m vectors of properties in D, $V_{mp}(A^{(p)}) = m^p V_m(A)$. In particular, for $p = 1$, an IIISSS of functions is said to be *homogeneous of degree one*.

Definition 7. A regular game $G = (N, V)$ whose IIISSS of functions $(V_m)_{m \geqslant 0}$ is homogeneous of degree p, is said to be *homogeneous of degree p*.

Example 3. Let $N = \{1, \ldots, n\}$ be a set of n $M/M/1$ queueing systems that cooperate in order to minimize the steady-state congestion in the combined system. Queueing system i is associated with its own exponential service rate μ_i and its own Poisson arrival rate of λ_i, $\lambda_i < \mu_i$, $i \in N$. Cooperation of a set $\varnothing \subseteq S \subseteq N$ results in a single $M/M/1$ queue whose service rate is $\mu(S) = \sum_{i \in S} \mu_i$, and whose arrival rate is $\lambda(S) = \sum_{i \in S} \lambda_i$. The cost associated with coalition S, $\varnothing \subseteq S \subseteq N$, is defined as the resulting mean number in the system. Let $G = (N, V)$ be the respective game where $V(S) = \lambda(S)/(\mu(S) - \lambda(S))$ for any S, $\varnothing \subseteq S \subseteq N$. This game was analyzed in Anily and Haviv (2010): the game is subadditive but it is not concave. The game is not formulated as a market game and it is not clear how to reduce it to a market game. Still it is proved in Anily and Haviv (2010) that the game is totally balanced and its nonnegative part of the core is fully characterized. In particular, it is shown that $\alpha_i = (\lambda_i / \sum_{j \in N} \lambda_j) V(N)$ is a core allocation. The game (N, V) is regular: each service provider $i \in N$ is assigned a vector of properties of size 2, namely, $(\lambda_i, \mu_i - \lambda_i)$ in $D = \Re^2_+$, where $(0, 0)$ is the null vector, and $V_1(0, 0) = 0$. The IIISSS of functions is given by $V_m((x_i, y_i)_{i=1\ldots m}) = \sum_{i=1}^m x_i / \sum_{i=1}^m y_i$. Regarding the homogeneity property defined above, it is easy to see that this game is homogeneous of degree zero.

Homogeneity of degree one means that when two (or more) identical sets of players cooperate, they cannot do better than what they did when acting individually. At the same time, none of them interfere with another. What they produce is just the total of what they would have produced separately. This in fact means constant return of scale. Note that subadditivity means that gains due to cooperation are possible. This, when coupled with homogeneity of degree one, means that in order to get a strict improvement, the cooperating sets should be different, i.e., at least one of the cooperating subsets should contain types of players that do not appear in the other set. In contrast to that, consider again the game presented in Example 3 and analyzed in Anily and Haviv (2010): this game is both subadditive and homogeneous of degree zero. Indeed, homogeneity of degree zero implies that when k identical coalitions cooperate the total cost is reduced by a factor of $1/k$. Thus, in this example both economies of scope and economies of scale prevail.

It is easy to verify the following:

Example 1 (Cont.). The $(k - 1) + 1$ sale game given in (3) with $D = \{(x_1, \ldots, x_k): x_j \geqslant 0 \; j = 1, \ldots, k\}$, is homogeneous of degree one.

Example 2 (Cont.). The location game given in (5) with $D = \{(x, y): (x, y) \in \Re^2 \times \Re^2\}$, is homogeneous of degree one.

Next we state our main theorem. The proof is deferred to §5.

Theorem 1. *Any regular game that is subadditive and homogeneous of degree one, is totally balanced.*

Theorem 1 is a new sufficient condition for total balancedness that helps us to resolve the question if the games in Examples 1 and 2 are totally balanced:

Example 1 (Cont.). The regular $(k - 1) + 1$ sale's game given in (3) is subadditive and homogeneous of degree one, and therefore it is totally balanced.

Example 2 (Cont.). The location game given in (5) is subadditive and homogeneous of degree one, and therefore it is totally balanced.

Theorem 1 provides a sufficient condition for total balancedness but it does not say how to generate cost allocations in the core for such games. This remains an open question.

5. Proof of Theorem 1

We start by reviewing a well-known necessary and sufficient condition for the nonemptiness of the core of a cooperative game; see, e.g., Osborne and Rubinstein (1994), chapter 13.

This condition is equivalent to the duality condition of a feasible linear programming formulation. Specifically, let $\mathscr{C}$ be the set of all 2^n coalitions of N. For any coalition S denote by $\Re^S$, the $|S|$-dimensional Euclidean space in which the dimensions are indexed by the members of S, and denote by $1_S \in R^n$ the characteristic vector of S given by

$$(1_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8. A collection $(\alpha_S)_{S\in\mathscr{C}}$ of numbers in $[0, 1]$ is said to be a *balanced collection of weights* if for every player $i \in S$ the sum of α_S over all coalitions that contain i equals 1, namely, $\Sigma_{S\ni i}\alpha_S = 1$ for all $i \in N$. A coalitional game $G = (N, V)$ is said to be balanced if $\sum_{S\in\mathscr{C}} \alpha_S V(S) \geqslant V(N)$, for every balanced collection of weights $(\alpha_S)_{S\in\mathscr{C}}$.

The following proposition is referred to as the Bondareva-Shapley Theorem; see, e.g., Proposition 262.1 in Osborne and Rubinstein (1994).

Proposition 1. *A coalitional game with transferrable utility has a nonempty core if and only if it is balanced.*

We are now ready to prove Theorem 1.

Proof. We prove the theorem by using Proposition 1 in two steps. We first prove that for any vector of balanced rational weights $(\alpha_S)_{S\in\mathscr{C}}$, the inequality $\sum_{S\in\mathscr{C}} \alpha_S V(S) \geqslant V(N)$, holds. Then we prove that the same is the case for any balanced collection of real weights.

Consider any balanced collection of rational weights $(\alpha_S)_{S\in\mathscr{C}}$. Let $M(\alpha)$ be a positive integer such that $\tau_S(\alpha) = M(\alpha)\alpha_S$ is an integer for all coalitions $S \in \mathscr{C}$. As the game $G = (N, V)$ is regular, there exists an integer $\kappa \geqslant 0$, such that each member $i \in N$ is associated with a vector of properties $y^i \in \Re^\kappa$. Let $y^i_j = y^i$ for any integer $j \geqslant 1$. Regularity of the game implies that $V(S) = V_{|S|}((y^i)|_{i\in S})$. As V is homogenous of degree one, $V_{\tau_S(\alpha)|S|}((y^i_j)_{(i,j)\in S^{(\tau_S(\alpha))}}) = \tau_S(\alpha)V(S)$. Note that

$$\sum_{S\in\mathscr{C}} \tau_S(\alpha)V(S) = \sum_{S\in\mathscr{C}} V_{\tau_S(\alpha)|S|}((y^i_j)_{(i,j)\in S^{(\tau_S(\alpha))}})$$
$$\geqslant V_{M(\alpha)n}((y^i_j)_{(i,j)\in N^{(M(\alpha))}}) = M(\alpha)V(N),$$

where the above inequality follows by the subadditivity of V in the regular game $G = (N, V)$, and specifically, subadditivity of V over $N^{(M(\alpha))}$ that contains $M(\alpha)$ repetitions of each player of N. Consider now the left-hand side of the inequality, i.e., $\sum_{S\in\mathscr{C}} V_{\tau_S(\alpha)|S|}((y^i_j)_{(i,j)\in S^{(\tau_S(\alpha))}})$: for any $i \in N$, we have also here $\sum_{S\in\mathscr{C}:\, i\in S} \tau_S(\alpha) = M(\alpha)\sum_{S\in\mathscr{C}:\, i\in S} \alpha_S = M(\alpha)$ copies of each vector of properties y^i, as $(\alpha_S)_{S\in\mathscr{C}}$ is a balanced collection of weights. The last equation follows from the fact that in the regular game $G = (N, V)$, the characteristic function V is homogenous of degree one. To conclude, $\sum_{S\in\mathscr{C}} \tau_S(\alpha)V(S) \geqslant M(\alpha)V(N)$. Recall that $\tau_S(\alpha) = M(\alpha)\alpha_S$, thus dividing the last inequality by $M(\alpha)$ gives the desired result for any rational balanced collection of weights $(\alpha_S)_{S\in\mathscr{C}}$.

To complete the proof, we need to show that the above property holds also for any vector of balanced real weights. Let $(\tilde{\alpha}_S)_{S\in\mathscr{C}}$, be a balanced collection of real weights. Consider the simplex induced by the constraints that define the set of balanced weights, i.e., $(\alpha_S)_{S\in\mathscr{C}} \geqslant 0$, and $\sum_{S\in\mathscr{C},\, i\in S} \alpha_S = 1$ for all $i \in N$. The extreme points of this simplex are rational, as the right-hand side of the constraints as well as the coefficients of the variables are 0 or 1. Let K be the number of extreme points of this simplex, and let α^j for $j = 1, \ldots, K$, be the respective extreme points, where each α^j is a vector of size $|\mathscr{C}|$. Thus, $(\tilde{\alpha}_S)_{S\in\mathscr{C}}$, can be represented as a convex combination of the extreme points: let $(\gamma_1, \ldots, \gamma_K)$ be the respective weights, so that $0 \leqslant \gamma_i \leqslant 1$ for $i = 1, \ldots, K$, $\sum_{j=1}^K \gamma_j = 1$, and $(\tilde{\alpha}_S)_{S\in\mathscr{C}} = \sum_{j=1}^K \gamma_j(\alpha^j_S)_{S\in\mathscr{C}}$. As each of the extreme points of the simplex is rational and is a vector of balanced weights, we have $\sum_{S\in\mathscr{C}} \alpha^j_S V(S) \geqslant V(N)$ for all $1 \leqslant j \leqslant K$. Therefore, $\sum_{S\in\mathscr{C}}(\tilde{\alpha}_S)V(S) = \sum_{S\in\mathscr{C}}\sum_{j=1}^K \gamma_j\alpha^j_S V(S) = \sum_{j=1}^K \gamma_j \sum_{S\in\mathscr{C}} \alpha^j_S V(S) \geqslant \sum_{j=1}^K \gamma_j V(N) = V(N)$.

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