Multi-Item Replenishment and Storage Problem (MIRSP): Heuristics and Bounds
Author(s): Shoshana Anily
Reviewed work(s):
Source: *Operations Research*, Vol. 39, No. 2 (Mar. - Apr., 1991), pp. 233-243
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/171100
Accessed: 26/02/2012 03:29

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

INFORMS is collaborating with JSTOR to digitize, preserve and extend access to *Operations Research.*

http://www.jstor.org

MULTI-ITEM REPLENISHMENT AND STORAGE PROBLEM (MIRSP): HEURISTICS AND BOUNDS

SHOSHANA ANILY

Tel Aviv University, Tel Aviv, Israel

(Received November 1988; revisions received June, November 1989; accepted November 1989)

Automated warehouses are often faced with the problem of smoothing their stock volume over time in order to minimize the cost due to space acquisition. In this paper, we consider an infinite-horizon, multi-item replenishment problem: In addition to the usual setup and holding costs incurred by each item, an extra charge proportional to the peak stock volume at the warehouse is due. This last cost raises the need for careful coordination while making decisions on the individual item order policies. We restrict ourselves to the class of policies that follows a stationary rule for each item separately. We derive a lower bound on the optimal average cost over all policies in this class. Then we investigate the worst case of the Rotation Cycle policy. We show that depending on the problem's parameters, the Rotation Cycle policy may yield an extremely good solution but in other settings this heuristic may generate an extremely poor policy. We also develop a new heuristic whose performance is at least as good as that of the Rotation Cycle procedure, and moreover, it is guaranteed to come, independently of the problem's parameters, within no more than 41% of the optimal solution!

In many distribution systems significant expenses are incurred by storage facilities, such as warehouses or depots. This cost rate, in the case of leasing the storage facility, usually depends on the room size required for holding the products. The well developed techniques of automated warehouses are based on a computerized system which controls both the storage and retrieval operations. This modernization allows for an integrated room allocation, i.e., the space allocation is not determined for each item separately—instead, the items share a common space consisting of multipurpose storage bins; such bins can store different products at different points of time.

In this paper, we consider a multi-item replenishment and storage problem (MIRSP) in the infinite horizon where all cost parameters and demand rates are constant, item-dependent but stationary over time. Backlogging is not allowed. The model is an extension of the EOQ model where, in addition to the traditional setup and inventory holding costs, a payment is incurred for the storage space required for holding the stock in the warehouse. The storage cost is assumed to be proportional to the maximum total stock volume held at the warehouse at one point in time, or equivalently, to the minimum warehouse size required for storing the items. This cost component ties the items together and raises the need for a careful coordination while making the decisions on the item-order quantities, on one hand, and the replenishment epochs phasing, on the other. The determination of the peak storage requirement may be extremely complicated even if each item follows an order policy which is characterized by a single constant order quantity. Therefore, we focus here on the derivation of a tight lower bound on the optimal average cost as well as the development of heuristics with small worst case bounds.

This problem has some similarity to the well known Economic Lot Scheduling Problem (ELSP) where n items are to be produced on a single machine; the machine can produce one item at a time. As in the EOQ model, the ELSP cost structure involves a setup cost and an inventory holding cost for each item separately. However, the replenishments of different items should be coordinated simultaneously because of the feasibility constraints, i.e., each item is produced at a certain speed and the problem is to find an optimal feasible production schedule (minimizing the total average cost) such that all demands are met on time.

The ELSP has received considerable attention. Many heuristics have been developed usually with little knowledge, if any, of their quality. A partial list of references includes Dobson (1986, 1987), Elmagraby (1978), Goyal (1975), Maxwell and Singh (1983) and Schweitzer and Silver (1983). Inman and Jones (1987) reported on a worst case analysis for the simplest of all heuristics called the Rotation Cycle (RC), where all items share a common replenishment interval. The authors compare the performance of the RC with the known lower bound on the optimal solution called the Independent Solution (IS). For each item i, $i = 1, \ldots, n$ let δ_i denote the ratio between the ith item cost components, namely the setup

Subject classifications: Analysis of algorithms: worst case bounds. Inventory/production: storage space requirement in automated warehouses. Inventory/production, deterministic models: EOQ with annual linear cost for storage space.

© 1991 Operations Research Society of America

cost/unit holding cost. Also define $\delta_{\min}$ and $\delta_{\max}$ to be the smallest and the greatest such ratios, respectively. The authors bound the gap between the RC and the IS solutions by a simple expression that depends solely on the ratio between $\delta_{\max}$ and $\delta_{\min}$. Their analysis leads to the observation that in many cases the Rotation Cycle policy provides an *almost* optimal policy. For example, if $\delta_{\max}$ is less than two times $\delta_{\min}$, then the RC is guaranteed to come within 4% of a lower bound on the optimal solution! Anily and Federgruen (1991b) consider the problem of multimachine ELSP where all machines are identical and work in parallel. They develop extremely simple heuristics which are shown to provide nearly optimal schedules both by a worst case bound and an asymptotic analysis.

The similarity between these problems arises from the fact that both of them are special versions of the multi-item EOQ model which contain an additional complex component requiring the coordination of the order epochs and order quantities of the different items. However, the source of complexity and therefore the solution methods of the two problems are very different.

It is well known that for the multi-item replenishment problem, in which the cost structure is composed of setup costs and linear holding costs, the EOQ formula can be invoked for each item separately resulting in an optimal policy that satisfies the following properties:

1. Zero Inventory Ordering (ZIO): an item is ordered only when its inventory is depleted (drops to zero).
2. Stationarity Between Orders (SBO): all quantity orders for any single item are of equal size.

In the sequel we show that the property of ZIO holds for the MIRSP as well. On the other hand, we construct a counterexample that demonstrates that the SBO property is not true in general. However, in view of the simplicity of implementing rules using constant order quantities for each item separately and the tremendous difficulty involved in the control of the peak storage requirement for general policies we limit ourselves a priori to policies satisfying the SBO property.

In contrast to the ELSP, the MIRSP has received much less attention by researchers in spite of its large applicability in automated warehouses, where one is often interested in smoothing the stock volume over time. For a description of the problem see Hodgson and Howe (1982). Park and Yun (1985) considered a discrete time scheduling of periodic tasks over an infinite horizon where the objective is to minimize the peak work load required. Hall (1988) proposed a simple heuristic for the MIRSP where all items share a common replenishment interval (similar to the ELSP's RC). For simplicity's sake we call the policy proposed by Hall the Rotation Cycle (RC) policy for the MIRSP. Hall provides, first, a detailed schedule of all the items—order epochs in a given order interval, and second, the order interval value that brings the overall average cost to a minimum. Hariga (1988), in his extensive work, provides some solution techniques for the same problem and some of its variants. The proposed solution methods, which are based both on exact formulations and heuristics, are not shown to exhibit any ex-ante bound on the optimality gap.

This paper investigates the worst case behavior of the RC heuristic for the MIRSP. We show, similarly to the worst case bound obtained for the ELSP's RC (see Inman and Jones), that the worst case gap of the MIRSP's RC is also a function of the problem's parameters: If the products have similar characteristics, then the RC has a high potential of being a very good solution, possibly the optimal one. However, in the presence of a large variability among the products' characteristics, the RC strategy may perform extremely poorly. Fortunately, we could derive an alternative heuristic, the Dynamic Rotation Cycle (DRC), for the MIRSP whose average cost is at least as low as that of the RC and in any event its worst case gap is bounded by the constant $\sqrt{2} = 141\%$ independently of the problem parameters!

We conclude this section with an overview of the paper. In Section 1, we provide some notation and preliminaries. In Section 2, we develop a lower bound for the average system-wide costs over all policies satisfying the SBO property. This lower bound involves the derivation of a good lower bound on the peak stock volume for any given order policy in that class. In Section 3, we evaluate the effectiveness of the RC suggested by Hall by comparing its average cost to the lower bound proposed in Section 2. In Section 4, we propose the Dynamic Rotation Cycle (DRC) heuristic which performs, at least as well as the RC, and moreover, its worst case bound cannot exceed $\sqrt{2} = 1.41$.

1. NOTATION AND PRELIMINARIES

Let

n = the number of different items in the system;
K_i = the setup cost for ordering item i; suppose that $K_i > 0$ for $i = 1, \ldots, n$;
D_i = the demand rate for item i;
h_i = the unit holding cost per unit of time of item i; and define $H_i = h_i D_i$;
s_i = the volume in feet^3 of item i; let
$S_i = D_i s_i$ and $S = \sum_{i=1}^{n} S_i$. S_i represents the consumption rate in feet^3 of items i where S represents the total consumption rate in feet^3;

w = the cost per unit of time for acquiring 1 foot^3 as storage space at the warehouse. Without loss of generality we assume that $w = 1$.

Let $I_i(t)$ represent the number of units of items i that are stored at the warehouse at time t.

As explained in the Introduction, the storage cost at the warehouse per unit of time grows linearly with the maximum stock volume held at the warehouse over time. Next, we show that there exists an optimal policy satisfying the Zero Inventory Ordering (ZIO) property.

Claim 1. *For any MIRSP there exists an optimal replenishment policy which satisfies the ZIO property. Also, optimal SBO policies are ZIO.*

Proof. Assume by contradiction that the claim is false, that is, there exists an MIRSP instance for which all the optimal policies do not satisfy the ZIO property. Among all the optimal strategies choose an arbitrary policy Q. For each item i define the sequence of time epochs τ_j^i $(j = 1, 2, \ldots) \tau_1^i < \tau_2^i < \cdots$ such that τ_j^i = the jth time occurrence of an order for item i, according to policy Q, while its inventory has not been depleted yet (i.e., $I_i(\tau_j^{i-}) > 0$).

Define a new policy Q' which is obtained as follows: policy Q' is a copy of Q except that the orders for item i which are placed at $t = \tau_1^i, \tau_2^i, \ldots, \tau_l^i, \ldots$ will be delayed to $\tau_j^i + I_i(\tau_j^{i-})/D_i$, $j = 1, 2, \ldots$, $i = 1, \ldots, n$. Once can easily check that Q' satisfies the ZIO property although all three cost components: setup costs, holding costs and storage requirement cost are not increased by this modification. The new policy obtained is thus also optimal, which contradicts our assumption. The second part of the claim regarding the SBO policies follows directly from this proof.

The next example demonstrates that there does not necessarily exist an optimal policy which satisfies the (SBO) property.

Example 1. Our example consists of two items $\{1, 2\}$ with $K_1 = 576$, $K_2 = 0.2$, $h_1 = h_2 = 0$, $S_1 = 4$, $S_2 = 1$. First we compute the best policy that satisfies the ZIO and the SBO properties. Let $T_1(T_2)$ be the replenishment interval of item 1 (2). Let also $V(T_1, T_2)$ denote the optimal average cost of such a policy. The average setup and holding cost of the policy is therefore given by

$$\frac{K_1}{T_1} + \frac{K_2}{T_2} + \frac{1}{2}H_1T_1 + \frac{1}{2}H_2T_2 = \frac{K_1}{T_1} + \frac{K_2}{T_2}$$

since $H_1 = H_2 = 0$.

Without loss of generality suppose that an order of item 1 is placed at time 0 and let $\tilde{t} > 0$ be the first point of time that an order for item 2 is placed ($\tilde{t} < T_2$). Let also $C(t)$ and C^* denote the stock volume (in feet^3) at time t and the storage space requirement, respectively, i.e., $C^* = \sup_{0 \leqslant t < \infty} C(t)$. Then $C(0) = 4T_1 + \tilde{t}$ and $C(\tilde{t}) = 4T_1 - 4\tilde{t} + T_2$. Note that $\min\max\{C(0), C(\tilde{t})\}$ is obtained for $\tilde{t} = T_2/5$ and, moreover, $C(0) = C(T_2/5) = 4T_1 + T_2/5$. Obviously, $C^* \geqslant 4T_1 + T_2/5$, therefore

$$V(T_1, T_2) \geqslant \frac{K_1}{T_1} + \frac{K_2}{T_2} + 4T_1 + \frac{T_2}{5}$$

$$= \frac{576}{T_1} + \frac{0.2}{T_2} + 4T_1 + \frac{T_2}{5}. \quad (1)$$

The right-hand side of (1) is minimized by $T_1^* = \sqrt{576/4} = 12$, $T_2^* = \sqrt{0.2/0.2} = 1$. Thus, $V^* \stackrel{\text{def}}{=} \inf_{T_1, T_2 > 0} V(T_1, T_2) \geqslant 96.4$. Morover, we will show that $V(12, 1) = 96.4$. Suppose that the orders for item 1, each for 48 feet^3, take place at $t = 0, 12, 24, \ldots$ and the orders for item 2, each for 1 feet^3, take place at $t = 0.2, 1.2, 2.2, \ldots$. The volume of the stock held at the warehouse does not exceed 48.2 feet^3 and it reaches this level at $t = 0, 0.2, 12, 12.2 \ldots$. Thus, $V(12, 1) \leqslant \frac{576}{12} + \frac{0.2}{1} + 48.2 = 96.4$, which implies that

$$V^* = V(12, 1) = 96.4.$$

Consider another policy that satisfies the ZIO but not the SBO property: order item 1 at $t = 0, 12, 24, \ldots$ each time for a quantity of 48 feet^3. However, for item 2 we do not use an equidistant order policy. Instead, during a cycle of 12 units of time order 0.5 feet^3 of item 2 at the two replenishment epochs occurring just after an order of item 1 is placed, and at all other replenishment epochs of item 2 let the order size be 1 foot^3. More precisely, item 2 is ordered at $t = 0.1, 0.6, 1.6, 2.6, \ldots, 10.6, 11.6, 12.1, 12.6, 13.6, \ldots$. We note that during a cycle of 12 time units, item 2 is ordered 13 times. The peaks in the storage requirement occur at $t = 0, 0.1, 12, 12.1, \ldots$ each for 48.1 feet^3. Thus, the average cost of this policy, which does not satisfy the SBO property, is

$$576/12 + 13.0 \cdot 2/12 + 48.1 \approx 96.32 < 96.4$$

showing in fact that no optimal policy satisfies the SBO property.

For the reason stated in the Introduction we restrict ourselves to the class of policies satisfying the SBO property even though this class is not guaranteed to contain the optimal strategy (see Example 1). Let Φ = {all replenishment policies satisfying the SBO property}. Also, let T_i denote the replenishment interval of item i where $X_i = S_i T_i$ represents its order quantity in feet^3.

We complete this section with the following lemma which was proved by Inman and Jones in the context of the ELSP. This lemma is used in the continuation to establish the worst case gaps of the heuristics discussed below.

Lemma 1. *Given a sequence* $\{(a_i, b_i)\}_{i=1}^n$ $a_i > 0$, $b_i > 0$, $i = 1, \ldots, n$ *such that* $a_1/b_1 \leqslant a_2/b_2 \leqslant \cdots \leqslant a_n/b_n$ *then*

$$\left(\sum_{i=1}^n a_i \sum_{i=1}^n b_i\right)^{1/2} \Big/ \sum_{i=1}^n (a_i b_i)^{1/2} \leqslant \left(1 + \frac{(1-\lambda^{1/2})^2}{2\lambda}\right)^{1/2}$$

where $\lambda \overset{\text{def}}{=} (a_1/b_1)/(a_n/b_n)$.

2. LOWER BOUND

In this section, we derive a simple lower bound for the peak stock volume of a given policy in the class Φ: We first derive the lower bound for a subclass of policies in Φ, namely the class $\Phi_2 \subseteq \Phi$ of power of two policies. Observe that the total inventory volume at the warehouse of policies in the class Φ_2 follows a cyclic pattern, and thus is much simpler to control than that of general policies in Φ. Later we show that the same lower bound remains valid for general policies in Φ. The expression obtained is then used to calculate a simple lower bound on the optimal average system-wide costs for all policies in Φ.

The following definitions will be used.

Definition 1. The **Inventory Level Graph** (ILG) is a graph representing the total stock volume at the warehouse of a given order policy Q (not necessarily in Φ) as a function of the time t. Let $C_Q(t)$ denote the corresponding function.

One can easily verify that $C_Q(t)$ is a piecewise linear function, right continuous and decreasing at a constant rate of $-S = -\sum_{i=1}^n S_i$. The points of discontinuity correspond to the replenishment epochs at which the total inventory volume is raised by a certain amount.

Definition 2. The **Multi-Item Replenishment Graph** (MIRG) for $(T_1, \ldots, T_n)$ is an ILG for an order policy of the items $\{1, \ldots, n\}$ in the class Φ such that $X_j = S_j T_j$ feet^3 of item j are ordered at equidistant intervals of T_j time units. The sequence of order intervals $(T_1, T_2, \ldots, T_n)$ does not uniquely determine the MIRG. The form of the MIRG depends on the replenishment epochs phasing as well as the order quantities.

For a given sequence of order intervals $T_1, T_2, \ldots, T_n$ let $P(T_1, \ldots, T_n) \overset{\text{def}}{=} \{$all replenishment policies in Φ that order $X_i = S_i T_i$ feet^3 of item i every T_i time units $i = 1, \ldots, n\}$ and $Y(T_1, \ldots, T_n) \overset{\text{def}}{=} \inf\{$peak stock volume (in feet^3) as encountered by policy

$$Q \mid Q \in P(T_1, \ldots, T_n)\}. \tag{2}$$

Observe that $Y(T_1, \ldots, T_n)$ represents the lowest storage space requirement associated with the MIRGs corresponding to $(T_1, \ldots, T_n)$.

Suppose that the sequence $(T_1, \ldots, T_n)$ is a power of two sequence, i.e., $T_i = \beta 2^{k_i}$, $k_i \in Z$ (Z = the set of integers) and $\beta > 0$. Also let $T^* = \max_{1 \leqslant i \leqslant n} T_i$. Clearly, the total inventory volume under a policy using the order intervals $(T_1, \ldots, T_n)$ follows a periodic pattern with a cycle length T^*. Moreover, the area below the MIRG for a time interval $[t, t+T^*)$, for any $t > 0$, equals $T^* \sum_{i=1}^n X_i/2$. Let $m_i = T^*/T_i$ denote the number of orders for item i placed during $[0, T^*)$. First we wish to bound from below the value $Y(T_1, \ldots, T_n)$, namely, the minimum storage space required by a policy in $P(T_1, \ldots, T_n)$. In order to do that we consider a broader set of policies $-P'(T_1, \ldots, T_n)$, which contains $P(T_1, \ldots, T_n)$ as a subset: $P'(T_1, \ldots, T_n)$ consists of all the periodic policies with a cycle length T^* for which the inventory level at the warehouse is not allowed to drop below zero, but the inventory level of each item separately can be negative. Moreover, we require that the area below the ILG during one cycle equals $T^* \sum_{i=1}^n X_i/2$ and exactly m_i orders of item i are placed during a cycle; each is for X_i feet^3, $i = 1, \ldots, n$. The inventory level graphs associated with the policies in P' do not necessarily use equal order intervals for each item; however, they satisfy these properties:

1. they are positive piecewise linear decreasing at a constant slope $-S$;
2. they follow a cyclic pattern with a cycle length of T^*;
3. the area under the curve during one full cycle is $T^* \sum_{i=1}^n X_i/2$;
4. the accumulated jump size (total increment) of the graph during one cycle of T^* time units equal to $\sum_{i=1}^n m_i X_i$ feet^3.

We also define $\underline{Y}(T_1, \ldots, T_n) = \inf\{$peak stock volume (in feet^3) of policy $Q \mid Q \in P'(T_1, \ldots, T_n)\}$. Obviously, $\underline{Y}(T_1, \ldots, T_n) \leqslant Y(T_1, \ldots, T_n)$. We say that an ILG corresponding to policy Q^*, $Q^* \in P'(T_1, \ldots, T_n)$ is *optimal* if $C^*_{Q^*} \overset{\text{def}}{=} \sup_{t \geqslant 0} C_{Q^*}(t) = \inf\{\sup_{t \geqslant 0} C_Q(t) \mid Q \in P'(T_1, \ldots, T_n)\}$.

Next we analyze the form of the optimal ILGs corresponding to policies in P', i.e., the graphs that bring the peak stock volume to a minimum: The next claim shows that these are the ones that smooth the total stock volume over time. In other words, any local maxima of the ILG is also a global maxima.

Claim 2. *Let $Q \in P'(T_1, \ldots, T_n)$ correspond to an optimal ILG for the power of two sequence $(T_1, \ldots, T_n)$, that is*

$$C_Q^* = \max_{0 \leqslant t \leqslant T^*} C_Q(t) = \underline{Y}(T_1, \ldots, T_n).$$

Then:

a. *each order point of the policy Q corresponds to an order of a single item l, $1 \leqslant l \leqslant n$ for a quantity of X_l feet^3;*
b. *the total stock volume at the warehouse according to policy Q reaches the same level at all replenishment epochs.*

Proof. Suppose by contradiction that there exists at least one optimal ILG associated with policy $Q \in P'(T_1, \ldots, T_n)$ which does not satisfy the claim. We need to distinguish between two cases: 1) According to Q at time t_k, $t_k < T^*$ at least two orders take place; one order is for X_l feet^3 of item l and the other is for X_k feet^3 of item k. In that case define $t_l = t_k$. 2) According to Q, suppose that at $t_k \leqslant T^*$, the function of $C_Q(\cdot)$ reaches a global maxima and, moreover, the local maxima preceding the one at t_k is not a global maxima. Let t_l, $t_l < t_k$ be the occurrence time of the order preceding the one at t_k, thus $C_Q(t_l) < C_Q(t_k)$. Similar to the previous case, assume that X_k feet^3 of item k are ordered at t_k.

In both cases, consider the modification of the policy Q: first, delay the orders for item k that occur at $t = t_k$ modulo (T^*) to $t = (t_k + \Delta)$ modulo (T^*) and second, advance the orders for item l that occur at $t = t_l$ modulo (T^*) to $t = t_l - \varepsilon$ modulo (T^*) for $\varepsilon = X_k \Delta / X_l$. Moreover, choose Δ small enough such that

a. according to Q, no orders take place in $[t_l - \varepsilon, t_l) \cup (t_k, t_k + \Delta]$, and

b. $\Delta < X_l / S - (t_k - t_l) X_l / X_k$. (It is easily verified that the right-hand side of the last inequality is positive in both cases.)

Let Q' be the new policy. The choice of Δ and ε ensures that the area under $C_{Q'}(t)$ over a full cycle T^* is unchanged by the modification (see Figure 1). Also

$$C_Q(t) = C_{Q'}(t)$$
$$\text{for any } t \notin [t_l - \varepsilon, t_l) \cup [t_k, t_k + \Delta) (\text{modulo } T^*).$$

If $C_{Q'}(\cdot)$ obtains a local maxima at t_l or t_k, then obviously its value is smaller than $C_Q(t_k)$. The two new peaks in the total stock volume of Q' during one full

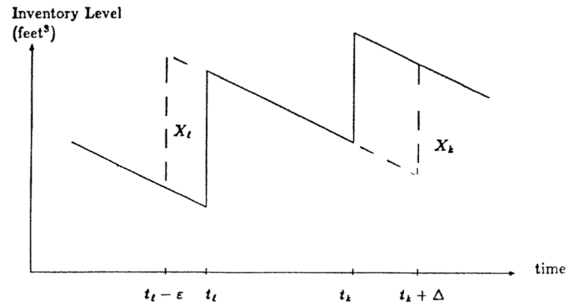

Figure 1. The bold lines denote the total stock volume according to Q where the dotted lines show the change in that quantity according to Q'. (Suppose that at $t_l(t_k)$ a single order takes place. A similar graph also holds for the general case.)

cycle occur at $t_l - \varepsilon$ and $t_k + \Delta$; however

$$C_{Q'}(t_k + \Delta) = C_Q(t_k) - S\Delta < C_Q(t_k)$$
$$C_{Q'}(t_l - \varepsilon) = C_{Q'}(t_k) + (t_k - t_l + \varepsilon) S$$
$$< C_{Q'}(t_k) + X_k = C_Q(t_k)$$

where the strict inequality follows from $\varepsilon = X_k \Delta / X_l$ and the choice of Δ. It is also easy to check that the total area under the graph during a cycle of T^* time units is not affected by this modification.

If the new policy Q' does not contradict our assumption that Q is optimal by requiring a smaller stock volume peak than Q', then the same procedure can be repeated on Q' until a contradiction is obtained.

We can proceed with the exact specification of the optimal ILGs associated with policies in $P'(T_1, \ldots, T_n)$. For that purpose we define $\Delta_i = X_i / S$ and recall that $m_i = T^* / T_i$.

Theorem 1. *Given a power of two sequence $(T_1, \ldots, T_n)$*:

a. $\sum_{i=1}^n m_i \Delta_i = T^*$.
b. *Let Q be an optimal policy, with respect to the storage space requirement, in $P'(T_1, \ldots, T_n)$ and suppose that $t_1 < t_2$ are two consecutive order points of Q such that an order for X_l feet^3 of item l is placed at t_2. Then $t_2 - t_1 = X_l / S = \Delta_l$.*
c. $\underline{Y}(T_1, \ldots, T_n) = \frac{1}{2} \sum_{i=1}^n S_i T_i + \frac{1}{2} S_i^2 T_i / S$.
d. $Y(T_1, \ldots, T_n) \geqslant \underline{Y}(T_1, \ldots, T_n)$.

Proof

a. $$\sum_{i=1}^n m_i \Delta_i = \sum_{i=1}^n \frac{T^*}{T_i} \cdot \frac{X_i}{S} = \frac{T^*}{S} \sum_{i=1}^n \frac{S_i T_i}{T_i} = T^*.$$

b. In view of Claim 2, $C_Q(t_1) = C_Q(t_2)$, but $C_Q(t_2) = C_Q(t_1) - (t_2 - t_1)S + X_l$ which imply that $t_2 - t_1 = X_l / s = \Delta_l$.

c. In view of parts a and b, optimal policies in P' are obtained by dividing $[0, T^*)$ into $\sum_{i=1}^{n} m_i$ subintervals; m_i of them are of length Δ_i. (According to part a the collection of these subintervals covers $[0, T^*)$.) At each right end-point of an interval Δ_i, an order is placed for X_i feet^3 of item i. The area under the corresponding set of ILGs during $[0, T^*)$ is obtained by subtracting the areas of the $\sum_{i=1}^{n} m_i$ triangles, m_i of them with base Δ_i and height X_i, $i = 1, \ldots, n$, from the area of the rectangle with height $\underline{Y} = \underline{Y}(T_1, \ldots, T_n)$ and base T^*, (see Figure 2).

Therefore

$$\underline{Y}T^* - \sum_{i=1}^{n} m_i \Delta_i X_i / 2 = T^* \sum_{i=1}^{n} X_i / 2$$

which implies that

$$\underline{Y}T^* - \frac{1}{2} \sum_{i=1}^{n} \frac{T^*}{T_i} \frac{X_i^2}{s} = T^* \sum_{i=1}^{n} X_i / 2$$

or equivalently

$$\underline{Y} = \frac{1}{2} \sum X_i + \frac{1}{2} \sum_{i=1}^{n} X_i S_i / S$$

$$= \frac{1}{2} \sum_{i=1}^{n} S_i T_i + \frac{1}{2} \sum_{i=1}^{n} S_i^2 T_i / S.$$

d. This follows directly from the fact that $P \subseteq P'$ and $Y(\underline{Y})$ is the lowest storage requirement for the policies in $P(P')$, thus $Y(T_1, \ldots, T_n) \geqslant \underline{Y}(T_1, \ldots, T_n)$.

The extension of the lower bound on the storage space requirement for general policies satisfying the ZIO and SBO properties, but not necessarily powers of two, requires a more careful analysis because of the complications involved in the control of the peak stock volume which may be acyclic and may not reach a maximum level.

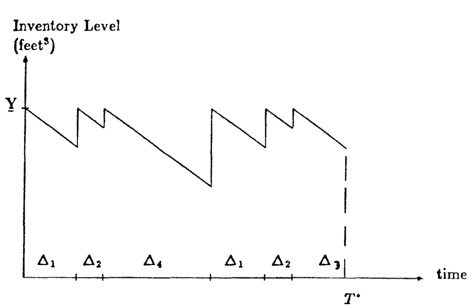

Figure 2. An optimal ILG in $P'(T_1, T_2, T_3, T_4)$ where $\Delta_i = X_i / S$, $m_1 = m_2 = 2$, $m_3 = m_4 = 1$.

Theorem 2. *The peak stock volume (= storage requirement) over the infinite horizon of any policy in Φ that follows the sequence of order intervals $(T_1, \ldots, T_n)$ is bounded from below by*

$$\underline{Y}(T_1, \ldots, T_n) = \frac{1}{2} \sum_{i=1}^{n} S_i T_i + \frac{1}{2} \sum_{i=1}^{n} S_i^2 T_i / S.$$

Proof. Suppose that Q is an optimal policy that satisfies the conditions stated in the theorem. Let $U = \sup_{t \geqslant 0} C_Q(t)$. (For simplicity we write $C(\cdot)$ instead of $C_Q(\cdot)$.) Without loss of generality we can assume that $U - \varepsilon < C(0) \leqslant U$ for a given $\varepsilon > 0$. Moreover, because of the optimality of Q, any policy that is obtained from Q by truncation of an initial part must require a storage space of U feet^3. Thus, we can find an infinite sequence of replenishment epochs $0 = \tau_0' < \tau_1' < \tau_2' \ldots$ such that $U - \varepsilon < C(\tau_i') \leqslant U$ and $C(\tau_{i-1}') \leqslant C(\tau_i')$, $i = 1, 2, \ldots$.

Define a new sequence of epochs:

$$\tau_0 = 0,\ \tau_i = \tau_i' + \frac{C(\tau_i') - C(0)}{S} \quad i = 1, 2, \ldots$$

then $\tau_i' \leqslant \tau_i + \varepsilon / S$.

By choosing $\varepsilon < \min_{1 \leqslant i \leqslant n} X_i / 2$ we guarantee that if $\tau_i > \tau_i'$, then no order is placed in the intervals $(\tau_i', \tau_i]$. Therefore

$$C(\tau_i) = C(\tau_i') - S(\tau_i - \tau_i') = C(0) \quad i = 0, 1, \ldots.$$

Let $m_i(t) =$ the number of orders for item i placed in $[0, t]$. Clearly

$$m_i(t) \geqslant \lfloor t / T_i \rfloor \tag{3}$$

and

$$C(t) = C(0) + \sum_{i=1}^{n} m_i(t) X_i - St$$

which implies that for $t = \tau_k$, k $= 1, 2, \ldots,$ $0 = C(\tau_k) - \mathrm{C}(0) = \sum_{i=1}^{n} m_i(\tau_k) \mathrm{X}_i - S\tau_k$, or equivalently, that

$$\sum_{i=1}^{n} m_i(\tau_k) X_i / S = \sum_{i=1}^{n} m_i(\tau_k) \Delta_i = \tau_k. \tag{4}$$

(The result in (4) is the analog to the one in Theorem 1, part a for power of two policies.) Let $A(t)$ denote the area under the given MIRG during $[0, t]$. Observe that there are at least $\lfloor t / T_i \rfloor$ full cycles of length T_i each, for item i's stock during $[0, t]$. Thus

$$A(t) \geqslant \frac{1}{2} \sum_{i=1}^{n} \lfloor t / T_i \rfloor X_i T_i \geqslant \frac{1}{2} \sum_{i=1}^{n} \left(\frac{t}{T_i} - 1 \right) X_i T_i$$

$$= \frac{t}{2} \sum_{i=1}^{n} X_i - \frac{1}{2} \sum_{i=1}^{n} X_i T_i$$

$$= \frac{t}{2} \sum_{i=1}^{n} S_i T_i - \frac{1}{2} \sum_{i=1}^{n} S_i T_i^2. \tag{5}$$

In light of (3), (4) and (5) we derive a lower bound $\underline{Y}_{\tau_k}(T_1, \ldots, T_n)$ on the peak stock volume during $[0, \tau_k]$ $k = 1, 2, \ldots,$ for all policies in Φ using the cycle sequence $(T_1, \ldots, T_n)$. The overall lower bound on the peak stock volume during $[0, \infty)$ will then be obtained by calculating $\lim_{k\to\infty} \underline{Y}_{\tau_k}(T_1, \ldots, T_n)$.

In view of (4) the total demand in $[0, \tau_k]$ equals the total order volume during the same period. A similar trick as for the power of two policies is also used here: Instead of finding a lower bound on the storage space required by the MIRG corresponding to policy Q, we consider a larger set of graphs, namely all the ILGs that: 1) have exactly $m_i(\tau_k)$ jumps each for X_i feet^3 of item i during $[0, \tau_k]$, $i = 1, \ldots, n$, and 2) the area below the graph during $[0, \tau_k]$ equals $A(\tau_k)$. Following the same argument as the ones in the proofs of Claim 2 and Theorem 1, we write

$$\underline{Y}_{\tau_k}\tau_k - \frac{1}{2}\sum_{i=1}^{n} m_i(\tau_k) X_i \Delta_i = A(\tau_k)$$

which implies together with (3) and (5) that

$$\begin{aligned}
\underline{Y}_{\tau_k} &= \frac{1}{\tau_k}\left(\frac{1}{2}\sum_{i=1}^{n} m_i(\tau_k) X_i\Delta_i + A(\tau_k)\right) \\
&\geq \frac{1}{\tau_k}\left(\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\tau_k}{T_i} - 1\right) X_i\Delta_i \right.\\
&\qquad\left. + \frac{\tau_k}{2}\sum_{i=1}^{n} S_iT_i - \frac{1}{2}\sum_{i=1}^{n} S_iT_i^2\right) \\
&= \frac{1}{\tau_k}\left(\frac{\tau_k}{2S}\sum_{i=1}^{n} S_i^2T_i - \frac{1}{2S}\sum_{i=1}^{n} S_i^2T_i^2\right.\\
&\qquad\left. + \frac{\tau_k}{2}\sum_{i=1}^{n} S_iT_i - \frac{1}{2}\sum_{i=1}^{n} S_iT_i^2\right) \\
&= \frac{1}{2}\sum_{i=1}^{n} S_iT_i + \frac{1}{2S}\sum_{i=1}^{n} S_i^2T_i \\
&\qquad - \frac{1}{2\tau_k}\left(\frac{\sum S_i^2T_i^2}{S} + \sum S_iT_i^2\right).
\end{aligned}$$

Therefore, we conclude that

$$\lim_{k\to\infty} \underline{Y}_{\tau_k}(T_1, \ldots, T_n) \geq \frac{1}{2}\sum_{i=1}^{n} S_iT_i + \frac{1}{2}\sum_{i=1}^{n} S_i^2T_i/S$$

which implies that the right-hand side of the last inequality is a lower bound on the peak stock volume over the infinite horizon for all policies Φ corresponding to $(T_1, \ldots, T_n)$. Thus we define

$$\underline{Y}(T_1, \ldots, T_n) \overset{\text{def}}{=} \frac{1}{2}\sum_{i=1}^{n} S_iT_i + \frac{1}{2}\sum_{i=1}^{n} S_i^2T_i/S.$$

It is worth noting that if $T_1 = T_2 = \cdots = T_n = T^*$, then our lower bound $\underline{Y}$ on the peak stock volume coincides with the optimal storage requirement derived by Hall for that special case. Moreover, if the set of items consists of a single commodity $\underline{Y} = S_iT_i$, which is again tight with the storage requirement needed by that item when using an order interval of T_i time units.

We are ready to proceed with the derivation of the lower bound on the optimal average cost of all policies in Φ. Define V^* = the optimal average cost of all policies in Φ; and $V^*(T_1, \ldots, T_n)$ = the optimal average cost of all policies in Φ which order item i every T_i time units $i = 1, \ldots, n$. Thus

$$V^*(T_1, \ldots, T_n) = \sum_{i=1}^{n} K_i/T_i + \frac{1}{2}\sum_{i=1}^{n} H_iT_i + Y(T_1, \ldots, T_n)$$

where Y is given in (2). Therefore

$$\begin{aligned}
V^*(T_1, \ldots, T_n) &\geq \sum_{i=1}^{n} K_i/T_i + \frac{1}{2}\sum H_iT_i + \underline{Y}(T_1, \ldots, T_n) \\
&= \sum_{i=1}^{n} K_i/T_i + \frac{1}{2}\sum_{i=1}^{n} H_iT_i \\
&\qquad + \frac{1}{2}\sum_{i=1}^{n} S_iT_i + \frac{1}{2}\sum_{i=1}^{n} S_i^2T_i/S \qquad (6)
\end{aligned}$$

and

$$\begin{aligned}
V^* &= \inf_{T_1, \ldots, T_n} V^*(T_1, \ldots, T_n) \\
&\geq \min_{T_1, \ldots, T_n}\left\{\sum_{i=1}^{n} K_i/T_i + \frac{1}{2}\sum_{i=1}^{n}(H_i + S_i)T_i + \frac{1}{2}\sum_{i=1}^{n} S_i^2T_i/S\right\} \\
&= \sum_{i=1}^{n}\left(2K_i((H_i + S_i) + S_i^2/S)\right)^{1/2} \overset{\text{def}}{=} \underline{V}.
\end{aligned}$$

We conclude with the following theorem.

Theorem 3. *The average system-wide cost of any policy in* Φ *is bounded from below by*

$$\underline{V} = \sum_{i=1}^{n}\left[2K_i(h_i + S_i + S_i^2/S)\right]^{1/2}$$

i.e., $V^* \geq \underline{V}$.

3. WORST CASE ANALYSIS FOR THE RC POLICY

In this section we provide a worst case analysis for the *RC* heuristic proposed by Hall by comparing its performance to $\underline{V}$—the lower bound on the average system-wide costs obtained in the previous section. Let $V^{RC}(T)$ denote the average cost of a RC that uses a cycle length of T. Hall shows that

$$V^{RC}(T) = \sum_{i=1}^{n} K_i/T + \frac{1}{2}\sum_{i=1}^{n}(H_i+S_i)T + \frac{1}{2}\sum_{i=1}^{n} S_i^2 T/S.$$

Therefore, the optimal RC is obtained by setting the order interval to

$$T^{RC} = \left(2\sum_{i=1}^{n} K_i \bigg/ \left(\sum_{i=1}^{n}(H_i+S_i) + \sum_{i=1}^{n} S_i^2/S\right)\right)^{1/2}$$

and

$$V^{RC} \stackrel{\text{def}}{=} V^{RC}(T^{RC}) = \left(2\sum_{i=1}^{n} K_i\left(\sum_{i=1}^{n}(H_i+S_i) + \sum_{i=1}^{n} S_i^2/S\right)\right)^{1/2}.$$

Clearly, $V^{RC} \geqslant V^* \geqslant \underline{V}$. Below we bound the gap between V^{RC} and $\underline{V}$.

For each item i, $1 \leqslant i \leqslant n$, define $a_i = K_i$ and $b_i = (H_i + S_i) + S_i^2/S$. Without loss of generality we can assume that the items are numbered in ascending order of the ratios a_i/b_i, $i = 1, \ldots, n$. Define also $\lambda = (a_1/b_1)/(a_n/b_n)$; then by invoking Lemma 1 we conclude with the next theorem.

Theorem 4. *Given a group of items $\{1, \ldots, n\}$ numbered in ascending order of the ratios a_i/b_i where $a_i = K_i$, $b_i = H_i + S_i + S_i^2/S$, $i = 1, \ldots, n$, then*

$$\frac{V^{RC}}{\underline{V}} \leqslant \left(1 + \frac{(1-\sqrt{\lambda})^2}{2\lambda}\right)^{1/2} \stackrel{\text{def}}{=} f(\lambda) \qquad (7)$$

where $\lambda = (a_1/b_1)/(a_n/b_n)$.

Inman and Jones develop a similar bound for the ELSP; moreover, they analyze the behavior of the function $f(\lambda)$ as λ varies. Their main observation is that the function f changes at an extremely slow rate when λ is close to one. For example, if $\lambda = 1$, i.e., all ratios a_i/b_i are identical, $f(1) = 1$ which means that $V^{RC} = \underline{V}$, namely, the RC is the optimal solution. This fact is intuitively true as the value T^{RC} coincides with the optimal T_i minimizing (6), $i = 1, \ldots, n$. More surprising is the fact that $f(0.75) = 1.006$ or $f(0.5) = 1.042$, i.e., V^{RC} comes within 0.6% (4%) of the lower bound $\underline{V}$ for $\lambda = 0.75$ ($\lambda = 0.5$). However, if $\lambda = 0.1$, then the RC policy can only be guaranteed to come within 82% of the lower bound. Thus, for values of λ that are close to one practitioners should not hesitate when using the simple RC policy proposed by Hall. However, for small values of λ (say, $\lambda < 0.2$) the performance of other heuristics should be verified as alternatives to the RC policy. In the next section, we propose a new heuristic which, we believe, may provide a significantly better solution, especially for small values of λ.

4. THE DYNAMIC ROTATION CYCLE (DRC)

In this section, we propose a new heuristic called the Dynamic Rotation Cycle Policy (DRC) which is shown to perform at least as well as the RC. Moreover, we show below that the worst case gap of that heuristic is bounded by $\min\{f(\lambda), \sqrt{2}\}$ where λ and $f(\lambda)$ are defined in (7). It is worth noting that the RC policy may yield an extremely poor solution when $\lambda \downarrow 0$ as $\lim_{\lambda\downarrow 0} f(\lambda) = \infty$; thus, the worst case gap of the RC heuristic can be made arbitrarily large for values of λ which are sufficiently small. The worst case gap of the DRC, on the other hand, is uniformly bounded by the constant $\sqrt{2}$, in addition to the bound of $f(\lambda)$, whichever is smaller, i.e., the DRC policy is guaranteed to come within 41% of the optimal solution independently of the λ-value.

Recall that according to the RC policy all items share a common replenishment interval T^{RC} and the total inventory volume at the warehouse reaches the same level at all order epochs. Thus, by using the RC policy we may enjoy the benefit of smoothing the inventory at the warehouse at the expense of high average setup and holding costs for those items for which their EOQ order intervals deviate too much from T^{RC}—the actual order interval used by the RC strategy. According to the DRC policy the set of items $\{1, \ldots, n\}$ is partitioned into groups, such that items with *similar* cost parameters fall into the same group. In each group separately, we use an RC policy that smoothes the inventory volume associated with the items of that group. Recall that the RC policy is extremely effective when implemented on a set of similar items (λ is close to one). These RC policies are then combined together to obtain the DRC heuristic. We observe that the task of combining the rotation cycles of the groups is, in general, much easier than combining n different order intervals because the number of sets is usually much smaller than the number of items. This task may be further simplified by rounding the order intervals into powers of two. In the following, we suggest a method for partitioning the items into groups.

Let $\chi = \{W_1, \ldots, W_L\}$ be a partition of the set of items $W = \{1, \ldots, n)$, i.e., $\bigcup_{l=1}^{L} W_l = W$ and $W_i \cap W_j = \varnothing$, $1 \leqslant i < j \leqslant L$. Define $C(\chi)$ to be the optimal average cost of a strategy using the RC policy in each set W_l, $l = 1, \ldots, L$, separately. Recall that the order interval of the set W_l is given by

$$T^{RC}(W_l) = \left(2\sum_{i\in W_l} K_i \Bigg/ \left(\sum_{i\in W_l}(H_i+S_i) + \sum_{i\in W_l} S_i^2 \Bigg/ \sum_{i\in W_l} S_i\right)\right)^{1/2}.$$

The items in W_l are scheduled in cycles of $T^{RC}(W_l)$ time units according to the RC policy proposed by Hall. Therefore, the peak stock volume of the items in W_l equals

$$Y_l \stackrel{\text{def}}{=} \left(\sum_{i\in W_l} S_i + \sum_{i\in W_l} S_i^2 \Bigg/ \sum_{i\in W_l} S_i\right) T^{RC}(W_l).$$

Assuming that this space is acquired by the warehouse for storing the items in W_l and ignoring the fact that in automated warehouses the total space requirement for all items may be less than $\sum_{l=1}^{L} Y_l$, we write the average cost due to the items in W_l as

$$\left(2\sum_{i\in W_l} K_i \left(\sum_{i\in W_l}(H_i+S_i) + \sum_{i\in W_l} S_i^2 \Bigg/ \sum_{i\in W_l} S_i\right)\right)^{1/2}.$$

Therefore

$$C(\chi) \leqslant \sum_{l=1}^{L}\left(2\sum_{i\in W_l} K_i \sum_{i\in W_l}\left(H_i + S_i + S_i^2 \Bigg/ \sum_{i\in W_l} S_i\right)\right)^{1/2} \stackrel{\text{def}}{=} C'(\chi). \tag{8}$$

We also define problem **P**.

Problem P

Minimizes $\{C(\chi) \mid \chi$ is a partition of $W\}$. It is easily verified that for

i. $\chi = \{W\}$: $C(\chi) = C'(\chi) = V^{RC}$, and for
ii. $\chi = \{\{1\}, \{2\}, \ldots, \{n\}\}$: $C'(\chi) = \sum_{i=1}^{n}(2K_i(H_i+2S_i))^{1/2}$.

Case ii is similar to the Independent Solution (IS) in the ELSP context: Each item is scheduled on the machine in equidistant intervals where for each item separately the interval is determined by invoking the EOQ formula with the setup and holding costs of that item. Of course, in the ELSP there is no guarantee that the IS is a feasible schedule. However, it provides a lower bound on the average cost of all feasible policies. In the MIRSP, allocating space in the warehouse for each item separately means, as previously mentioned, that S_iT_i feet^3 should be reserved for item i. Therefore, the optimal cycle time for the group of items consisting of a single item i equals $(2K_i/(H_i+2S_i))^{1/2}$, $i = 1, \ldots, n$, and the average cost due to the group $\{\{i\}\}$ is given by $(2K_i(H_i+2S_i))^{1/2}$. However, note that implementing such a policy for each item in $W = \{1, \ldots, n\}$ may result in a total average cost which is smaller than the expression in (9) because in automated warehouses different items may share a common space. For simplicity we use the name the *Independent Solution* (IS) for this heuristic as well and denote the respective partition by X^{IS}, i.e., $X^{IS} = \{\{1\}, \ldots, \{n\}\}$ and

$$C(\chi^{IS}) \leqslant C'(\chi^{IS}) = \sum_{i=1}^{n}(2K_i(H_i+2S_i))^{1/2}.$$

The exact evaluation of $C(\chi)$ for an arbitrary partition χ of W involves the hard task of the determination of the peak stock volume at the warehouse over the infinite horizon. Therefore, instead of solving **P** we will focus on the solution of the RHS of (8) which we denote by **P′**.

Problem P′

Minimize $\{C'(\chi) : \chi$ is a partition of $W\}$. Alternatively, **P′** can be written as

$$\min\left\{\sum_{l=1}^{L}\left(2\sum_{i\in W_l} K_i \sum_{i\in W_l}\left(H_i + S_i + S_i^2 \Bigg/ \sum_{i\in W_l} S_i\right)\right)^{1/2} \,\middle|\, \chi = \{W_1, \ldots, W_L\} \text{ is a partition of } W\right\}.$$

Both **P** and **P′** are partitioning problems, i.e., a set of elements is to be partitioned into groups such that a certain cost function is optimized. General partitioning problems are known to be NP-hard; see Karp (1972). Efficient (polynomial) algorithms exist for very special forms of the cost function; see Chakravarty, Orlin and Rothblum (1982, 1985) and Anily and Federgruen (1991a). Barnes, Hoffman and Rothblum (1989) forms of cost functions; they obtain some nice characterizations of the geometrical aspects of the optimal partition; however, these are not sufficient yet for the development of efficient algorithms except for the cases discussed in Chakravarty, Orlin and Rothblum (1985).

Note that the group cost function in **P′** is separable in groups; however, the partitioning problem associated with it cannot be cast in any of the special structural partitioning problems which are known to be polynomially solvable. One cost function considered by Chakravarty, Orlin and Rothblum (1985) is similar to the one in P'; they consider the following partitioning problem (**PC**) described below.

Suppose that a set of elements $W = \{1, \ldots, n\}$ is to be partitioned into groups. Each element i is characterized by two attributes a_i and b_i $a_i \geqslant 0$, $b_i \geqslant 0$, $i = 1, \ldots, n$. Let $\chi = (W_1, \ldots, W_L\}$ be an arbitrary partition of W. Assume also that the group-cost function is a real valued function of the variables $\sum_{i\in W_l} a_i$ and $\sum_{i\in W_l} b_i$. Problem **PC** is given as follows.

Problem PC

Minimize

$$\left\{ \sum_{l=1}^{L} g\left(\sum_{i\in W_l} a_i, \sum_{i\in W_l} b_i \right) \mid \chi = \{W_1, \ldots, W_L\} \right.$$

$$\left. \text{is a partition of } W. \right\}$$

Lemma 2 provides some conditions under which **PC** can be solved in polynomial time. First we need the following definitions.

Definition 3. A set $W_l \subset W$ is said to be *consecutive* if the indices of its elements are consecutive integers, e.g., the set $\{2,4,5\}$ is not consecutive but $\{2,3,4,5\}$ is consecutive.

Definition 4. A partition $\chi = \{W_1, \ldots, W_L\}$ is *consecutive* if W_l is consecutive $l = 1, \ldots, L$.

Lemma 2. (Chakravarty, Orlin and Rothblum 1985) *Suppose that the group cost function $g(\cdot,\cdot)$ is jointly concave in both of its arguments. Then there exists an optimal consecutive partition for* **PC**.

The computation of an optimal consecutive partition can be accomplished by solving a shortest path algorithm in complexity $O(n_2)$.

Shortest Path Algorithm (SPA)

Let $G(n) = 0$ and

$$G(i) = \min_{i+1 \leqslant j \leqslant n} \left\{ g\left(\sum_{k=i+1}^{j} a_k, \sum_{k=i+1}^{j} b_k \right) + G(j) \right\}$$

$$i = n-1, \ldots, 0.$$

The optimal cost of a partition given by $G(0)$, as well as the optimal partition associated with it, can be obtained by solving recursively the above equations.

Unfortunately, $\mathbf{P}'$ does not satisfy all of the conditions in Lemma 2. The cost function in (8) appears to be separable in the groups and, moreover, the group cost function is jointly concave in the group-sum of two attributes:

1. K_i;
2. $(H_i + S_i + S_i^2 / \sum_{i\in W_l} S_i)$.

But the second attribute is not independent of the partition used. Thus, Lemma 2 cannot be implemented directly. However, based on Lemma 2 we will construct the DRC heuristic for the MIRSP which generates a good partition, although not necessarily the optimal one for $\mathbf{P}'$.

The Dynamic Rotation Cycle (DRC) Algorithm

Step 0. Number the items in ascending order of the ratios $K_i/(H_i + 2S_i)$, i.e., $K_1/(H_1 + 2S_1) \leqslant K_2/(H_2 + 2S_2) \leqslant \ldots$.

Step 1. (Solve the SPA)
$G(n) = 0$;
$i = n - 1$;
while $i \geqslant 0$ do
begin
$G(i) = \min_{i+1 \leqslant j \leqslant n}\{(2\sum_{k=i+1}^{j} K_k \sum_{k=i+1}^{j}(H_k + S_k + S_k^2/\sum_{m=i+1}^{j} S_m))^{1/2} + G(j)\}$;
$i = i - 1$;
end.

Clearly, during the execution of Step 1 one can also store the corresponding path, so that the optimal partition can be recovered. The optimal average cost of the partition generated by the DRC algorithm according to the cost function C' is given by $G(0)$. Let χ^{DRC} denote the generated partition. Then

$$C(\chi^{DRC}) \leqslant G(0) = C'(\chi^{DRC}).$$

It is also easily verified that both the partition associated with the *IS* and the one associated with the *RC* are feasible for the DRC algorithm. Therefore

$$C'(\chi^{DRC}) \leqslant \min\{C'(\chi^{RC}), C'(\chi^{IS})\}$$
$$= \min\{V^{RC}, C'(\chi^{IS})\}.$$

In the next theorem we investigate the worst case gap of the DRC algorithm.

Theorem 5

$$\frac{C'(\chi^{DRC})}{\underline{V}} \leqslant \min\{\sqrt{2}, f(\lambda)\}$$

where λ and $f(\lambda)$ are defined in (7).

Proof. In view of Theorem 4 and the fact that $C'(\chi^{DRC}) \leqslant V^{RC}$ we obtain the inequality

$$\frac{C'(\chi^{DRC})}{\underline{V}} \leqslant \frac{V^{RC}}{\underline{V}} \leqslant f(\lambda).$$

In addition, $C'(\chi^{DRC}) \leqslant C'(\chi^{IS}) = \sum_{i=1}^{n}(2K_i(H_i + 2S_i))^{1/2}$ which implies that

$$\frac{C'(\chi^{DRC})}{\underline{V}} \leqslant \frac{C'(\chi^{IS})}{\underline{V}}$$

$$= \frac{\sum_{i=1}^{n}(2K_i(H_i + 2S_i))^{1/2}}{\sum_{i=1}^{n}(2K_i(H_i + S_i + S_i^2/S))^{1/2}}$$

$$\leqslant \frac{\sum_{i=1}^{n}(2K_i(2H_i + 2S_i))^{1/2}}{\sum_{i=1}^{n}(2K_i(H_i + S_i))^{1/2}}$$

$$\leqslant \sqrt{2} = 1.41.$$

Note that for $\lambda = 0.18$, $f(\lambda) = \sqrt{2}$. Therefore, we can write the worst case gap of the DRC algorithm as

$$\frac{C'(\chi^{DRC})}{\underline{V}} \leqslant \begin{cases} f(\lambda) & \lambda \geqslant 0.18 \\ \sqrt{2} & \lambda < 0.18. \end{cases}$$

It is also worth noting that the worst case analysis was conducted on the basis of the two extreme heuristics, namely the IS and the *RC*. We can expect that in practice the DRC heuristic will provide a much better solution than these two. Moreover, we observe that the calculation of the function $C'(\chi)$ for a given partition χ of W assumes that the sets' order intervals are combined arbitrarily thus leading to a worst case storage space requirement (i.e., the sum of the sets' peak stock volume). In practice, if the optimal partition does not consist of a single set, the least sophisticated methods that one may use while combining the sets' order intervals will result in a space requirement that is smaller than the conservative one used by the function $C'(\cdot)$.

Given the optimal partition χ^*, the task of efficiently combining the sets' order intervals may be greatly simplified by rounding off these quantities into powers of two as described, for example, by Maxwell and Singh (1983) and Roundy (1985). They propose a rounding procedure of the sets' order intervals such that the new cost of each set is within 2%(!) of the original one. This minor increase in the cost may be offset later by the opportunity of saving on the space requirement at the warehouse: An ILG of a power of two policy follows a cyclic pattern and, therefore, it is sufficient to focus on a single cycle while combining the sets' order intervals into a policy.

ACKNOWLEDGMENT

This research has been supported by Natural Science Engineering Research Council Canada (NSERCC) grant A4802.

REFERENCES

ANILY, S., AND A. FEDERGRUEN. 1991a. Structured Partitioning Problems. *Opns. Res.* **39**, 130–149.

ANILY, S. AND A. FEDERGRUEN. 1991b. The Multi-Machine ELSP: Heuristics and Bounds (in preparation).

BARNES, E. R., A. J. HOFFMAN AND U. G. ROTHBLUM. 1989. Optimal Partitions Having Disjoint Convex and Conic Hulls. Working Paper, Faculty of Industrial Engineering and Management, Technion, Haifa, Israel.

CHAKRAVARTY, A. K., J. B. ORLIN AND U. G. ROTHBLUM. 1982. A Partitioning Problem With Additive Objective With an Application to Optimal Inventory Groupings for Joint Replenishment. *Opns. Res.* **30**, 1081–1020.

CHAKRAVARTY, A. K., J. B. ORLIN AND U. G. ROTHBLUM. 1985. Consecutive Optimizers for the Partitioning Problem With Applications to Optimal Inventory Groupings for Joint Replenishment. *Opns. Res.* **33**, 820–834.

DOBSON, G. 1986. The Cyclic Lot Scheduling Problem With Sequence Dependent Setups. Working Paper, Graduate School of Management, The University of Rochester, Rochester, N.Y.

DOBSON, G. 1987. The Economic Lot Scheduling Problem: A Resolution of Feasibility Using Time Varying Lot Sizes. *Opns. Res.* **35**, 764– 771.

ELMAGHRABY, S. E. 1978. The Economic Lot Scheduling Problem (ELSP): Review and Extensions. *Mgmt. Sci.* **24**, 587–598.

GOYAL, S. K. 1975. Scheduling a Multi-Product Single Machine System, A New Approach. *Intnl. J. Prod. Res.* **13**, 487–493.

HALL, N. G. 1988. A Multi-Item EOQ Model With Inventory Cycle Balancing. *Naval Res. Logis.* **35**, 319–325.

HARIGA, M. 1988. The Warehouse Scheduling Problem. Ph.D. Thesis, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, N.Y.

HODGSON, T. J., AND T. J. HOWE. 1982. Production Lot Sizing With Material-Handling Cost Considerations. *IIE Trans.* **14**, 44–51.

INMAN, R. R., AND P. C. JONES. 1987. When is the Economic Lot Scheduling Problem Easy? Technical Report 87-02, Center of Manufacturing Engineering, Northwestern University, Evanston, Ill.

KARP, R. M. 1972. Reducibility Among Combinatorial Problems. In *Complexity of Computer Computations*, R. E. Miller (ed.). Plenum Press, New York, 85–103.

MAXWELL, W. L., AND H. SINGH 1983. The Effect of Restricting Cycle Times in the Economic Lot Scheduling Problem. *IIE Trans.* **15**.

PARK, K. S., AND D. K. YUN. 1985. Optimal Scheduling of Periodic Activities. *Opns. Res.* **33**, 690–695.

ROUNDY, R. 1985. 98%—Effective Integer-Ratio Lot-Sizing for One Warehouse Multi-Retailer Systems. *Mgmt. Sci.* **31**, 1416–1430.

SCHWEITZER, P. J., AND E. A. SILVER. 1983. Mathematical Pitfalls in the One Machine Multiproduct Economic Lot Scheduling Problem. *Opns. Res.* **31**, 401–405.