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Single-machine lot-sizing with uniform yields and rigid demands: robustness of the optimal solution

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We consider the lot-sizing discrete problem of a single machine whose yield is uniformly distributed. We assume that the customers' demand is rigid, i.e. all demand must be satisfied. The costs involved are a setup cost paid each time a run is initiated plus a unit variable cost per unit produced. No salvage cost is associated with extra or defective units. We prove that there exists an optimal sequence of lot-sizes that is strictly increasing in the demand levels and therefore an optimal lot-size that is at least as large as the demand level. These properties have been proved in the literature only for binomial yields. We also provide an extremely simple algorithm to compute the optimal lot-sizes. Moreover, we show that the cost function is strongly robust in the lot-size: for any $\varepsilon > 0$ we develop a procedure that generates a (usually large) class of policies whose relative error is bounded by ε.

1. Introduction

This paper deals with the problem of determining the optimal lot-size for a single-machine production process with discrete random yield. We assume that demands are rigid, i.e. all of the quantity ordered must be supplied. Thus, for a given lot of size N, the yield (the number of non-defective items) is a random variable. If the realized yield is less than the current demand, an additional run will be required. We note that the yield may be determined only upon termination of the production of the whole lot.

The costs involved are a setup cost α paid each time a run is initiated plus a variable production cost β paid for each item produced. Defective items and extra good items are assumed to have no value. (Such a situation may arise in industries where the customer provides an exact and unique specification of the product configuration and the chance that successive customers will require a product with exactly the same characterizations is negligible.) To simplify notation we assume without loss of generality that $\beta = 1$: thus the monetary basic unit is chosen as the unit production cost.

The objective is to determine a production policy that minimizes the expected total cost. For a given demand D, a production policy is defined by a sequence of lot-sizes $\{N_d\}_{d=1}^{D}$. Lot-sizes that are too large may cause unnecessarily high production costs. On the other hand, too small lot-sizes may result in high costs due to frequent setups. Similar considerations arise while determining the optimal order size in procurement problems. In such models the uncertainty is related to the demand process. In single-period models one can assume that either a penalty cost for backlogging or shortages is due or, alternatively, a service-level constraint is imposed. The rigid demand environment corresponds to a 100% service-level constraint. The underlying assumption of random yield with rigid demand models is that the machine setups as well as the production process do not consume much time; thus a period is sufficiently long to ensure meeting the whole demand on time. Although procurement problems have been extensively studied and several types of model were investigated, only a few results are known about production lot-size with random yield, especially in the context of rigid demands.

In Section 2 we present the general recursive formula for general discrete yield distributions and we summarize the few known results in this field. In Section 3 we discuss and thoroughly analyze the uniform yield model. In Section 4 we show that the cost function is extremely robust in the lot-size, and producing non-optimal lots may cause a small increase in the expected cost. Section 5 concludes the paper with an experimental study that demonstrates the robustness of the cost function.

2. The general discrete formulation

The lot-sizing problem with general discrete yield distribution can be formulated by a set of recursive minimization problems. However, in the general case, a search must be conducted to solve each of the minimization problems. Such a search might be time-consuming and thus the whole procedure might be ineffective for large

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demand levels. Therefore it is extremely important to find structural properties of the optimal solution that may substantially reduce the number of different lot-sizes that should be scanned by the search procedure at each recursion step. We first present the basic notation and the general recursive formula:

D	demand level.
N	lot-size; N_D represents the optimal lot-size for demand level D.
Y_N	yield; a random variable that represents the yield out of a lot of size N.
$P(y, N) = P(Y_N = y)$	the probability of obtaining a yield of size y out of a lot of size N. It is natural to assume in this context that $P(y, N) = 0$ for $y > N$.
$V_D(N)$	expected cost to fulfill a demand of D units, if a lot of size N is produced whenever the outstanding demand is D and optimal lot-sizes are produced whenever the outstanding demand level is below D.
$V_D = \min\{V_D(N) : N \geq 1\}$	the optimal expected cost of fulfilling a demand for D units.

The recursive formula for general discrete models is

$$V_D(N) = \alpha + \beta N + P(0, N)V_D(N) + \sum_{y=1}^{D-1} P(y, N)V_{D-y},$$

or equivalently (taking $\beta = 1$),

$$V_D(N) = \frac{\alpha + N + \sum_{y=1}^{D-1} P(y, N)V_{D-y}}{1 - P(0, N)}. \quad (1)$$

Let also $N_D \in \operatorname{argmin}\{V_D(N) : N \geq 1\}$. As mentioned above, the main difficulty in computing N_D when no information about the function $V_D(N)$ is available is that essentially all $V_D(N)$ values for $N \geq 1$ up to an N that is sufficiently large must be computed. This paper investigates the structure of optimal policies in the uniform-yield case. The only additional non-trivial discrete yield distribution for which some structural properties of the optimal solution were proved is the binomial one, i.e.

$$P(y, N) = \binom{N}{y} p^y (1-p)^{N-y}, \quad 0 \leq y \leq N.$$

In this model each unit produced has a probability p to meet the requirements independently of the others. Indeed, Beja (1977) considers a more general model where the ratio between the unit variable cost and its success probability is constant. He proved that (1) N_D is strictly increasing in D, which also implies that $N_D \geq D$ and (2) that $V_D(N)$ is quasi-convex in N. As a result the search for N_D should be performed over $N > N_{D-1}$ up to the first integer for which $V_D(N) < V_D(N+1)$.

Beja (1977) and Grosfeld-Nir and Gerchak (1991) have noticed that general distributions fail to exhibit simple properties such as $N_D \geq D$ (the optimal lot-size is at least as large as the demand level) or $N_{D-1} \leq N_D$ (optimal lot-sizes are non-decreasing in the demand level). Beja (1977) has proved these properties for the binomial yield distribution indirectly, via a Markov decision model. Below we investigate yields that are uniformly distributed and we show that these properties hold also for the uniform yield model: there exists an optimal sequence of lot-sizes that are strictly increasing in the demand level. Moreover, *no search* is required in the evaluation of N_D; N_D is given by a simple expression of $V_1, V_2, \ldots, V_{D-1}$.

Grosfeld-Nir and Gerchak (1991) investigate the function V_D as a function of the setup cost parameter α for *general* discrete yield distributions and proved the next lemma (see Theorem 1 in their paper) which we use later. In order to express the dependence of $V_D(V_D(N))$ in α we may write instead $V_D(\alpha)(V_D(N, \alpha))$.

Lemma 1.
(a) the function $V_D(\alpha)$ is piecewise linear increasing and concave in α.
(b) the slope of $V_D(\alpha)$ at any α where the function is differentiable equals the expected number of setups for that α.

Another result that we use in the sequel appears in Theorem 2 in Grosfeld-Nir and Gerchak (1991); see Lemma 2 below. This lemma deals with general yield distributions for which the expected yield is proportional to the lot-size, as for example the binomial and the uniform distributions.

Lemma 2. If $\alpha = 0$ and for some constant ω, $0 < \omega \leq 1$, $E(Y_N) = \omega N \; \forall N$, then any lot of size N, $N \in \{1, \ldots, D\}$ is optimal for demand level D, and moreover $V_D = D/\omega$.

Additional relevant literature includes Klein (1966) and Sepheri *et al.* (1986), which provide heuristics for the binomial yield model. Two continuous yield models have also been studied, *both under the assumption that the lot-size is at least as large as the demand level.* (1) White (1965) developed a successive approximation algorithm for stochastic proportional yields, i.e., $Y_Q = QX$, where Q is the lot-size, Y_Q the respective yield and X is a random variable independent of Q that assumes values in between zero and one. (2) Grosfeld-Nir and Gerchak (1990) obtain an explicit solution for the uniform yield case.

For a thorough review on random yield and list of references see Yano and Lee (1995) and Grosfeld-Nir and Gerchak (1991).

3. The uniform model

The discrete uniform yield with $P(y, N) = 1/(N+1)$ for $0 \le y \le N$ is mentioned in Grosfeld-Nir and Gerchak (1991) and is explored herein. The binomial distribution may be viewed as the extreme case of a production process where the quality of one item is independent of the others. In the uniform distribution case the probability of one item's being defective does depend on the history of the process, i.e. the quality of the items previously produced in the same run. The uniform distribution possesses the following properties.

(1) The distribution is symmetric around the mean.

(2) $\sum_{y=k}^{N} P(y, N)$(= probability of obtaining at least k good items from a lot of size N) and $\sum_{y=0}^{N-k} P(y, N)$ (= probability of getting at least k defective items from a lot of size N) are increasing functions of N for any k, $0 \le k \le N$. In other words, the number of good and defective items tends to increase as the lot-size increases.

The following equation is obtained by substituting the uniform distribution into (1):

$$V_D(N) = \begin{cases} \alpha + 1 + \dfrac{\alpha + \sum_{i=D-N}^{D-1} V_i}{N} + N & \text{if } N < D-1, \quad \text{(2a)} \\[2ex] \alpha + 1 + \dfrac{\alpha + \sum_{i=1}^{D-1} V_i}{N} + N & \text{if } N \ge D-1. \quad \text{(2b)} \end{cases}$$

In Theorem 1 we prove that there exists a minimizer N_D of the function $V_D(N)$ that also minimizes the following function $f_D(N)$:

$$f_D(N) = \alpha + 1 + (\alpha + \sum_{i=1}^{D-1} V_i)/N + N, \quad N \ge 1, \quad (3)$$

and, moreover, $N_D \ge D$.

In view of the above results, N_D can be easily computed as follows: consider the function $g : \boldsymbol{N} \to \Re$ (where $\boldsymbol{N}$ denotes the set of positive integers) $g(N) = \theta/N + N$ and extend its definition over the positive real numbers so that $\tilde{g} : \Re^+ \to \Re$, $\tilde{g}(x) = \theta/x + x$. The function $\tilde{g}$ is convex and attains its minimum at $x^* = \sqrt{\theta}$. Also, $\tilde{g}(\gamma x^*) = \tilde{g}(x^*/\gamma) = \tilde{g}(x^*)/e(\gamma)$, where

$$e(\gamma) = 2/(\gamma + \gamma^{-1}) \quad (4)$$

(this function was also used by Roundy (1985)). The function $e(\gamma)$ is strictly quasi-concave, satisfies $e(\gamma) = e(\gamma^{-1})$ and achieves its maximum at $\gamma = 1$. From Roundy (1985) it follows that the unique integer N^* satisfying the following inequalities is an integer optimizer of the function g:

$$\sqrt{N^*(N^*-1)} \le x^* = \sqrt{\theta} < \sqrt{N^*(N^*+1)}.$$

(The function g may attain its minimum at two points if θ equals the product of two consecutive integers: each of these integers is a minimizer of the function g.)

Note that the unique integer satisfying

$$\sqrt{N^*(N^*-1)} \le \sqrt{\alpha + \sum_{i=1}^{D-1} V_i} < \sqrt{N^*(N^*+1)} \quad (5)$$

is an optimal solution for $f_D(N)$ (defined in (3)). If $f_D(N)$ is minimized at two points, then N^* is its greater minimizer. In this section we show that N^* is also a minimizer of $V_D(N)$. The following propositions will be helpful in proving Theorem 1.

Proposition 1. Suppose that N_D, a minimizer of $V_D(N)$, satisfies $N_D \ge D$; then

$$f_D(N_D) = \min\{f_D(N) \mid N \ge 1\},$$

where $f_D(N)$ is defined in (3).

Proof: Assume by contradiction that $N_D \ge D$ but for any minimizer G_D of $f_D(N)$, $G_D < D$. Then

$$\begin{aligned} V_D(G_D) &= \alpha + 1 + (\alpha + \sum_{i=D-G_D}^{D-1} V_i)/G_D + G_D \\ &\le \alpha + 1 + (\alpha + \sum_{i=1}^{D-1} V_i)/G_D + G_D \\ &= f_D(G_D) < f_D(N_D) = V_D(N_D) = V_D. \end{aligned}$$

(The weak inequality follows from V_i being non-negative, and for the strict inequality note that G_D is a global minimizer of f_D where N_D is not (according to our initial assumption). The third equality follows because the function $f_D(N)$ coincides with the function $V_D(N)$ for $N \ge D$.) Thus, in contradiction to the proposition's assumption, $G_D < D$ is a better lot-size than N_D for demand level D. We conclude that one of the minimizers of the function f_D is greater than or equal to D. Since the functions $f_D(N)$ and $V_D(N)$ coincide for $N \ge D-1$, N_D also minimizes f_D. ■

Proposition 2 states that if the optimal lot-size is not smaller than the demand level for two consecutive demands $D-1$ and D, then the respective optimal lots are strictly increasing in D.

Proposition 2. Let N_{D-1} and N_D be the greatest optimal lot-sizes for V_{D-1} and V_D respectively. If $N_{D-1} \ge D-1$ and $N_D \ge D$, then $N_D > N_{D-1}$.

Proof: In view of the fact that the functions $f_D(N)$ and

$V_D(N)$ coincide for $N \geq D-1$ then, according to (5) and Proposition 1, we obtain the following inequalities:

$$\alpha + \sum_{i=1}^{D-2} V_i \geq N_{D-1}(N_{D-1} - 1); \qquad \text{(a)}$$

$$\alpha + \sum_{i=1}^{D-1} V_i < N_D(N_D + 1); \qquad \text{(b)}$$

$$\left.\begin{aligned} N_D(N_D+1) &> \alpha + \sum_{i=1}^{D-1} V_i \geq N_{D-1}(N_{D-1}-1) + V_{D-1} \\ &= N_{D-1}(N_{D-1}-1) + \alpha + 1 + \\ &\quad \frac{\alpha + \sum_{i=1}^{D-2} V_i}{N_{D-1}} + N_{D-1} \\ &\geq N_{D-1}^2 + N_{D-1} + \alpha \geq N_{D-1}(N_{D-1}+1). \end{aligned}\right\} \qquad \text{(c)}$$

(In (c), the first inequality follows from (b) and the second and third from (a).) As a result we conclude that $N_D > N_{D-1}$. ■

The next proposition states that if the optimal lot-size is at least as large as the demand level for two consecutive demands $D-1$ and D, then increasing the demand from $D-1$ to D increases the optimal cost by at least two monetary units. Note that in order to increase the expected yield by a single unit, two additional units have to be produced, causing the total variable cost to increase by two.

Proposition 3. Under the assumptions of Proposition 2, $V_D - V_{D-1} \geq 2$.
Proof: In view of Proposition 2 and (2), $N_{D-1} \geq D-1$ and $N_D \geq D$ we obtain $N_D > N_{D-1}$ and

$$V_D - V_{D-1} = \frac{\alpha + \sum_{i=1}^{D-1} V_i}{N_D} + N_D - \left\{\frac{\alpha + \sum_{i=1}^{D-2} V_i}{N_{D-1}} + N_{D-1}\right\}.$$

In the proof we distinguish between the cases $N_D = N_{D-1} + 1$ and $N_D = N_{D-1} + \Delta$, $\Delta \geq 2$.

Case 1: it is sufficient to show that

$$(\alpha + \sum_{i=1}^{D-1} V_i)/N_D - (\alpha + \sum_{i=1}^{D-2} V_i)/(N_D - 1) \geq 1,$$

or, equivalently, that $(N_D - 1)V_{D-1} - (\alpha + \sum_{i=1}^{D-2} V_i) \geq N_D(N_D - 1)$. By expressing $\sum_{i=1}^{D-2} V_i$ in terms of V_{D-1} we obtain the desired result:

$$\begin{aligned} (N_D - 1)V_{D-1} - (\alpha + \sum_{i=1}^{D-2} V_i) &= (N_D - 1)V_{D-1} - (V_{D-1} - \alpha - 1 \\ &\quad - (N_D - 1))(N_D - 1) \\ &= (\alpha + 1 + N_D - 1)(N_D - 1) \geq \\ &\quad N_D(N_D - 1). \end{aligned}$$

Case 2: in that case it is sufficient to show that, for $\Delta \geq 2$,

$$(\alpha + \sum_{i=1}^{D-1} V_i)/N_D \geq (\alpha + \sum_{i=1}^{D-2} V_i)/(N_D - \Delta).$$

Here we use the following relations that hold according to the definitions of N_{D-1} and N_D:

$$\alpha + \sum_{i=1}^{D-2} V_i < N_{D-1}(N_{D-1} + 1) = (N_D - \Delta)(N_D - \Delta + 1)$$

and

$$\alpha + \sum_{i=1}^{D-1} V_i \geq N_D(N_D - 1).$$

Thus we conclude that

$$\begin{aligned} \frac{\alpha + \sum_{i=1}^{D-2} V_i}{N_D - \Delta} &< \frac{(N_D - \Delta)(N_D - \Delta + 1)}{N_D - \Delta} = N_D - \Delta + 1 \\ &\leq N_D - 1 = \frac{N_D(N_D - 1)}{N_D} \leq \frac{\alpha + \sum_{i=1}^{D-1} V_i}{N_D}. \end{aligned}$$

■

We are now prepared to prove the main theorem of the paper, which states that the optimal lot-size for the uniform yield case is no smaller than the demand level D and is therefore (see Proposition 2) also strictly increasing in D.

Theorem 1.
(a) $V_D(1) \geq V_D(2) \geq \ldots \geq V_D(D)$ *for* $D \geq 1$.
(b) *There exists an optimal sequence of lot-sizes* $\{N_D\}_{D \geq 1}$ such that $N_D \geq D$.
Proof: we first note that (b) is a direct consequence of (a). The proof is by induction on D. For $D = 1$ the theorem holds trivially. For $D = 2$ note that, in view of the definition of $V_D(N)$ in (2), $V_2(1) \geq V_2(2)$ if and only if $\alpha + V_1 \geq 2$, which holds trivially because $V_1 \geq 1 + N_1 \geq 2$. We shall prove that the theorem holds for $D \geq 3$. We now assume by induction that $V_i(k)$ is non-increasing for $k = 1, \ldots, i, i \leq D-1$ and therefore there exists an optimal sequence $N_i \geq i$ for $i \leq D-1$, where N_i is the greatest optimal lot-size for demand level i. We

shall prove that the theorem holds for $i = D$ as well. We first show that $V_D(k) \geq V_D(k+1)$ for $k = 1, \ldots, D-2$:

$$V_D(k) - V_D(k+1) \geq \frac{1}{k}\sum_{i=1}^{k} V_{D-i} - \frac{1}{k+1}\sum_{i=1}^{k+1} V_{D-i} - 1$$
$$= \frac{\sum_{i=1}^{k} V_{D-i} - kV_{D-k-1} - k(k+1)}{k(k+1)}.$$

(The first inequality follows because $\alpha/k \geq \alpha/(k+1)$.) It is sufficient to show that the numerator of the last expression is non-negative. But according to the induction's assumption and Proposition 3, $V_i - V_{i-1} \geq 2$ for $i \leq D-1$ thus we can write $V_{D-k+j} \geq 2(j+1) + V_{D-k-1}$ for $j = 0, 1, \ldots, k-1$. Therefore;

$$\sum_{i=1}^{k} V_{D-i} = \sum_{j=0}^{k-1} V_{D-k+j} \geq \sum_{j=0}^{k-1} 2(j+1) + kV_{D-k-1}$$
$$= 2\sum_{j=1}^{k} j + kV_{D-k-1} = k(k+1) + kV_{D-k-1}.$$

which proves that $V_D(1) \geq V_D(2) \geq \ldots \geq V_D(D-1)$. To terminate the proof it remains to show that $V_D(D-1) \geq V_D(D)$. However, for $k = D-1$ and $k = D$,

$$V_D(k) = \alpha + 1 + (\alpha + \sum_{i=1}^{D-1} V_i)/k + k. \quad (6)$$

Note that according to Lemma 1 the functions V_i are piecewise linear increasing in α, and therefore by (6) $V_D(D-1)$ and $V_D(D)$ also have these same properties. From (6) it is easy to see that if both $V_D(D-1)$ and $V_D(D)$ are differentiable in α then

$$\frac{\partial}{\partial \alpha} V_D(D-1) \geq \frac{\partial}{\partial \alpha} V_D(D).$$

In view of Lemma 2 at $\alpha = 0$ both functions coincide, i.e. $V_D(D-1) = V_D(D)$. Since both functions are piecewise linear increasing in α with the first having a slope at least as large as the second at any α where the functions are differentiable, we conclude that $V_D(D-1) \geq V_D(D)$ for any $\alpha \geq 0$. Thus there exists an optimal lot-size N_D for demand level D such that $N_D \geq D$. ■

The next theorem summarizes the results of Theorem 1 and Propositions 1,2,3:

Theorem 2.
(a) *There exists an optimal strictly increasing sequence of lot-sizes* $\{N_D\}_{D \geq 1}$.
(b) *The optimal expected cost* V_D *can be calculated recursively from:*

$$V_D = \min_{N \geq 1}\left\{\alpha + 1 + (\alpha + \sum_{i=1}^{D-1} V_i)/N + N\right\}.$$

(c) *An optimal strictly increasing sequence* $\{N_D\}_{D \geq 1}$ *can be obtained by the inequalities*

$$N_D(N_D - 1) \leq \alpha + \sum_{i=1}^{D-1} V_i < N_D(N_D + 1).$$

This sequence defines the sequence of the greatest optimal lot-sizes.

Next we present a recursive algorithm for computing the sequence of greatest optimal lot-sizes $\{N_i\}_{1 \leq i \leq D}$. To simplify the presentation, when saying that θ is rounded to the 'nearest' integer we mean that θ is rounded to $\lfloor \theta \rfloor$ if $\theta/\lfloor \theta \rfloor < \lceil \theta \rceil/\theta$ and θ is rounded to $\lceil \theta \rceil$ otherwise. (By convention the 'nearest' integer to 0 is 1.) As can be easily verified, the algorithm's complexity is linear in D with an extremely small coefficient.

Algorithm OPT-LS

begin

  $V_0 := 0$;
  For $i = 1, \ldots, D$ do begin
    round $(\alpha + \sum_{j=1}^{i-1} V_j)^{0.5}$ to the 'nearest' integer N_i;
    $V_i := \alpha + 1 + (\alpha + \sum_{j=1}^{i-1} V_j)/N_i + N_i$;
  end ;
end.

4. Intensitivity of the cost function for non-optimal lots

In this section we shall show that the single machine lot-sizing problem with uniform yields and rigid demand is robust in the sense that reasonable deviations from the optimal lot-sizes at each demand level may cause only a small increase in the total expected costs. In this section we shall present the magnitude of these deviations.

The analysis of the cost robustness in the lot-size is important because usually there exist other considerations beside the expected production set-up and variable costs that were modeled explicitly: for example, it is quite common that certain raw materials are purchased in batches, and breaking such a batch may cause damage to the remaining material (chemical products). On the other hand, it may happen that the amount of raw materials at the plant is insufficient for producing optimal lot-sizes: precious time may be wasted while waiting for the raw material to arrive. The manager can greatly improve his/her decisions by evaluating the increase in the expected cost if production is started immediately with the on-hand raw materials, in comparison with the cost of shutting down the machine or/and delaying the supply date.

In the sequel we present, for each $\varepsilon > 0$, a procedure that generates *a family of policies* with a relative error bounded by ε. The relative error of a policy is a measure of its effectiveness and is defined as follows: for a given problem let V represent the optimal expected cost and V^H the expected cost of a specific policy denoted by H; the *relative error of policy H*, e^H, is defined as

$$e^H = (V^H - V)/V.$$

In this section, we provide, for any given $\varepsilon > 0$, a procedure that generates a family of policies defined by a sequence of pairs of integers $(L_d(\varepsilon), U_d(\varepsilon)), d \geq 1$: given an initial demand D, any sequence of integers $\{N_d^H\}_{1 \leq d \leq D}$, satisfying $L_d(\varepsilon) \leq N_d^H \leq U_d(\varepsilon)$, defines a policy in this family that produces a lot of size N_d^H whenever the remaining demand level is d. The relative error of all these policies is bounded by ε, i.e. the expected cost of any of these policies does not exceed the optimal cost by more than $100\varepsilon\%$.

As mentioned earlier (see algorithm OPT-LS), N_D is obtained by rounding $(\alpha + \sum_{i=1}^{D-1} V_i)^{0.5}$, which we denote by X_D, to the 'nearest' integer. Thus $X_D(N_D)$ is the continuous (integer) minimizer of $V_D(N), N \geq D$. Let $\tilde{V}_D = \alpha + 1 + 2X_D$. Obviously $\tilde{V}_D$ is the minimum of the function $V_D(N)$ when the integrality constraint is relaxed; thus it is a lower bound for the optimal expected cost for demand D, i.e. $\tilde{V}_D \leq V_D$. In our analysis we compare the expected cost of the policies generated by the procedure with either the optimal expected cost or with the lower bound $\tilde{V}_D$. In all cases we show that $V_D^H/V_D \leq 1 + \varepsilon$.

We first describe how to generate the sequence of pairs $(L_D(\varepsilon), U_D(\varepsilon)), d \geq 1$, for a given $\varepsilon > 0$. Note that the infinite sequence of pairs depends only on ε and α and is independent of the initial demand D. For a given D it is necessary to compute only the first D pairs of the sequence.

4.1. *A procedure for generating a $(1+\varepsilon)$ family of policies*

Step 0. Let $X_d(\varepsilon) := (\alpha + (1+\varepsilon)\sum_{i=1}^{d-1} V_i)^{0.5}$ $1 \leq d \leq D$;
$\gamma_1(\varepsilon) := 1 + \varepsilon + (2\varepsilon + \varepsilon^2)^{0.5}$, and
$\gamma_d(\varepsilon) := (1+\varepsilon)^{0.5} + \varepsilon^{0.5}, \quad 2 \leq d \leq D.$

Step 1: For $d = 1, \ldots, D$ calculate $L_d(\varepsilon)$ and $U_d(\varepsilon)$ as follows:
If $N_d \in [X_d(\varepsilon)/\gamma_d(\varepsilon), \gamma_d(\varepsilon)X_d(\varepsilon)]$ then $L_d(\varepsilon) := \lceil X_d(\varepsilon)/\gamma_d(\varepsilon) \rceil$ and $U_d(\varepsilon) := \lfloor \gamma_d(\varepsilon)X_d(\varepsilon) \rfloor$;
If $N_d < X_d(\varepsilon)/\gamma_d(\varepsilon)$ then $L_d(\varepsilon) := N_d$ and $U_d(\varepsilon) := \lfloor \gamma_d(\varepsilon)X_d(\varepsilon) \rfloor$;
Otherwise, $L_d(\varepsilon) := N_d$ and $U_d(\varepsilon) := N_d$.

Note that the above-defined procedure may allow lot-sizes smaller than the demand level; see also Section 5. Observe also that in the case that $N_D > \gamma_D(\varepsilon)X_D(\varepsilon)$ then, since N_D is the 'nearest' integer to X_D and $X_D < X_D(\varepsilon)$, it must hold that N_D is also the 'nearest' integer to $X_D(\varepsilon)$. Thus, since the 'nearest' integer to $X_D(\varepsilon)$, N_D does not belong to the interval $[X_d(\varepsilon)/\gamma_d(\varepsilon), \gamma_d(\varepsilon)X_d(\varepsilon)]$; this interval must therefore be empty of integers. Below we prove that the above family of policies has a relative error bounded by ε.

Theorem 3. The relative error of any policy H that is defined by a sequence of lot-sizes N_d^H whenever the remaining demand level is d, $d \geq 1, N_d^H \in [L_d(\varepsilon), U_d(\varepsilon)]$, is bounded by ε.

Proof: Let H be a specific policy that employs a lot-size N_i^H whenever the remaining demand is i and assume that H satisfies the theorem conditions. Let $V_d^H = V_d^H(N_1^H, N_2^H, \ldots, N_d^H)$ be the expected cost of this policy if the initial demand level is $d, d \geq 1$. We shall prove by induction that $V_d^H/V_d \leq 1 + \varepsilon$ for $d \geq 1$.

For $d = 1$ note that since $X_1 = X_1(\varepsilon) = \alpha^{0.5}$ and N_1 is the 'nearest' integer to X_1, then either the interval $[X_1/\gamma_1(\varepsilon), \gamma_1(\varepsilon)X_1]$ contains N_1 or the interval is empty of integers, thus $L_1(\varepsilon) = U_1(\varepsilon) = N_1$. The second case is trivial because only the optimal lot-size is allowed. To prove the first case, let N_1^H be an integer in the interval $[X_1/\gamma_1(\varepsilon), \gamma_1(\varepsilon)X_1(\varepsilon)]$, then

$$\frac{V_1^H}{V_1} \leq \frac{V_1(N_1^H)}{\tilde{V}_1} = \frac{\alpha + 1 + \alpha/N_1^H + N_1^H}{\alpha + 1 + 2X_1}$$
$$\leq \frac{\alpha + 1 + 2X_1/e(\gamma_1(\varepsilon))}{\alpha + 1 + 2X_1} < (e(\gamma_1(\varepsilon)))^{-1} = 1 + \varepsilon.$$

(See (4) for definition of the function $e(\ )$. The second inequality follows from the fact that X_1 is the continuous minimizer of $\alpha/X + X$ and $\alpha/X_1 + X_1 = 2X_1$; also N_1^H satisfies $\gamma_1^{-1}(\varepsilon) \leq N_1^H/X_1 \leq \gamma_1(\varepsilon)$; thus $\alpha/N_1^H + N_1^H \leq 2X_1/e(\gamma_1(\varepsilon))$. The last inequality follows from the fact that the maximum value that the function $e(\ )$ can assume is 1. The last equality follows from the definition of $\gamma_1(\varepsilon)$ that satisfies the equality $e(\gamma_1(\varepsilon)) = (1+\varepsilon)^{-1}$.)

Suppose now, by induction, that for any demand level $d, d \leq D - 1$, the relative error of the policies in the above family is bounded by ε: we shall prove correctness also for $d = D$. Thus, according to the induction's assumption, for any specific policy H in the above family

$$V_d^H(N_1^H, N_2^H, \ldots, N_d^H) \leq (1+\varepsilon)V_d \quad 1 \leq d \leq D - 1.$$

If $N_D^H \in [X_D(\varepsilon)/\gamma_D(\varepsilon), \gamma_D(\varepsilon)X_D(\varepsilon)], D \geq 2$, then

$$\frac{V_D^H}{V_D} \leq \left[\alpha + 1 + \frac{\alpha + \sum_{d=1}^{D-1} V_d^H}{N_D^H} + N_D^H\right] / \tilde{V}_D$$

$$\leq \left[\alpha + 1 + \frac{\alpha + (1+\varepsilon)\sum_{d=1}^{D-1} V_d}{N_D^H} + N_D^H\right] / \tilde{V}_D$$

$$\leq \left[\alpha + 1 + \frac{2\sqrt{\alpha + (1+\varepsilon)\sum_{d=1}^{D-1} V_d}}{e(\gamma_D(\varepsilon))}\right] / \tilde{V}_D$$

$$\leq \left[\alpha + 1 + 2(1+\varepsilon)\sqrt{\alpha + \sum_{d=1}^{D-1} V_d}\right] / \tilde{V}_D < 1 + \varepsilon.$$

(The first inequality follows from the definition of V_D^H, and $\tilde{V}_D \leq V_D$, and it holds also for the case that $N_D^H < D$. The second inequality follows from the induction's assumption about V_d^H, $d \leq D-1$. The third inequality follows from the fact that $x = X_D(\varepsilon)$ is the continuous minimizer of the following function $(\alpha + (1+\varepsilon)\sum_{d=1}^{D-1} V_d)/x + x$ and $\gamma_D(\varepsilon)^{-1} \leq N_D^H/X_D(\varepsilon) \leq \gamma_D(\varepsilon)$. The fourth inequality follows from the choice of $\gamma_D(\varepsilon)$, $D \geq 2$, for which $e(\gamma_D(\varepsilon)) = (1+\varepsilon)^{-0.5}$, and for the last inequality use the definition of $\tilde{V}_D$.)

In particular for a policy H with N_d^H, $d \leq D-1$, satisfying the theorem's conditions and $N_D^H = N_D$:

$$\frac{V_D^H(N_1^H, \ldots, N_{D-1}^H, N_D)}{V_D} \leq \frac{\alpha + 1 + (\alpha + \sum_{d=1}^{D-1} V_d^H)/N_D + N_D}{V_D}$$

$$\leq \frac{\alpha + 1 + (\alpha + (1+\varepsilon)\sum_{d=1}^{D-1} V_d)/N_D + N_D}{\alpha + 1 + (\alpha + \sum_{d=1}^{D-1} V_d)/N_D + N_D} < 1 + \varepsilon.$$

The only case that remains to be considered is if $N_D < \lceil X_D(\varepsilon)/\gamma_D(\varepsilon)\rceil$ and the interval $[X_d(\varepsilon)/\gamma_d(\varepsilon), \gamma_d(\varepsilon)X_d(\varepsilon)]$ is non-empty: we have to show that for any N_D^H such that $N_D < N_D^H < \lceil X_D(\varepsilon)/\gamma_D(\varepsilon)\rceil$ $V_D^H(N_1^H, N_2^H, \ldots, N_D^H) < (1+\varepsilon)V_D$. However, it is easily seen that the function $V_D^H(N_1^H, N_2^H, \ldots, N_{D-1}^H, x)$ is convex in x and thus the set of real numbers for which the function value is bounded by the constant $(1+\varepsilon)V_D$ is convex. Since the inequality holds for N_D and the integers in the interval, then it must hold for all integers in between. ■

In the next section we shall show that even for relatively small ε, the robustness property allows a lot of flexibility in the selection of the lot sizes. The magnitude of the width of the intervals is approximately determined by the values $\gamma_1(\varepsilon)$ and $\gamma_d(\varepsilon)$ for $d \geq 2$, which are given in Table 1 for some selected values of ε.

Table 1. $\gamma_1(\epsilon)$ and $\gamma_d(\epsilon) d \geq 2$ for some selected values of ϵ.

ϵ	$\gamma_1(\epsilon)$	$\gamma_d(\epsilon); d \geq 2$
0.01	1.152	1.105
0.02	1.221	1.151
0.03	1.227	1.189
0.04	1.326	1.220
0.05	1.370	1.250
0.06	1.412	1.275
0.1	1.558	1.365
0.2	1.863	1.542

For example, for $\varepsilon = 0.05$ and $D = 500$ suppose that $X_D(\varepsilon) = 700$: all integers in the interval $[700/1.25, 700 \times 1.25] = [560, 875]$ are legitimated lot-sizes for the first run.

5. Experimental study

In this section we report the results of an experimental study that demonstrates the problem's robustness. For $\alpha = 1$, 50, 500 (note that the unit variable cost was assumed to be 1), and ε values 1%, 5% and 10% we report the optimal, the minimum and maximum lot-size for $d = 1, 2, \ldots, 10$ and $d = 20, 30, \ldots, 100$ in Tables 2, 3 and 4 respectively.

As can be observed from the tables (and can be shown from the algorithm definition) for any specific demand level and α the intervals $[L_D^H, U_D^H]$ are nested; the interval for small ε is contained in the intervals for larger ε. Also for given demand level and ε, the larger α is, the larger the interval is. For given ε and α notice that the procedure may provide a huge number of policies, which increases as α and/or ε increase. For example, for $D = 10$, $\varepsilon = 5\%$ and $\alpha = 1$, the procedure generates 8640 different policies. For $D = 10$, $\alpha = 500$ and $\varepsilon = 1\%$ we obtain 13 948 526 592 different policies!

It is also important to note that the policies generated by the procedure for a given ε have a relative error that is *at most* ε. However, usually the effective relative error is much smaller. For example, consider the problem with $D = 10$ and $\alpha = 50$. The optimal sequence of lot-sizes $(N_1, N_2, \ldots, N_{10}) = (7, 11, 14, 16, 19, 21, 23, 25, 27, 29)$ and $V_{10} = 108.7622$. The following policy falls in the class of $\varepsilon = 5\%$ $(N_1^H, N_2^H, \ldots, N_{10}^H) = (9, 9, 17, 20, 16, 18, 19, 31, 34, 36)$. It is easy to see that the lot-sizes of this policy are the lower or the upper bound of the intervals allowed, so we would expect its expected cost to be relatively high. A simple calculation shows that $V_{10}^H = 110.3742$, which is less than 1.5% above the optimal expected cost.

Table 2. $\alpha = 1$

D	N_D	L_D^H, U_D^H		
		$\varepsilon = 1\%$	$\varepsilon = 5\%$	$\varepsilon = 10\%$
1	1	1, 1	1, 1	1, 1
2	2	2, 2	2, 2	2, 3
3	3	3, 3	3, 4	3, 4
4	5	5, 5	4, 5	4, 6
5	6	6, 6	5, 7	5, 8
6	7	7, 7	6, 8	6, 9
7	8	8, 8	7, 9	6, 11
8	9	9, 9	8, 11	7, 12
9	10	9, 10	9, 12	8, 14
10	11	10, 12	9, 13	9, 15
20	21	20, 23	18, 27	17, 30
30	31	29, 34	26, 40	25, 44
40	42	38, 46	35, 53	32, 59
50	52	47, 57	43, 66	40, 73
60	62	57, 68	51, 78	48, 88
70	72	66, 79	59, 91	56, 102
80	82	75, 90	68, 104	63, 117
90	92	84, 102	76, 117	71, 131
100	102	93, 113	84, 130	79, 146

6. Summary and conclusions

This paper analyzes the optimal lot-sizing of a single-machine production process with discrete uniform yield under rigid demand. For any given demand level we develop a linear time algorithm that generates the optimal policy. In particular we show that the structural properties of the optimal lot-sizing, proved for the reject and allowance model, hold also for the uniform yield: (1) there is a sequence of optimal lot-sizes $\{N_D\}_{D \geq 1}$ that is strictly increasing in D and therefore also $N_D \geq D$ for $D \geq 1$; (2) the cost function is quasi-convex in the lot-size. These two properties allow for an effective search of the optimal lot-size for any given demand level.

We hope that the analysis in this paper will motivate the research for deriving general conditions on the yield distribution under which the above two properties hold. Grosfeld and Nir (1991) state a conjecture regarding this

Table 3. $\alpha = 50$

D	N_D	L_D^H, U_D^H		
		$\varepsilon = 1\%$	$\varepsilon = 5\%$	$\varepsilon = 10\%$
1	7	7, 8	6, 9	5, 11
2	11	10, 11	9, 13	9, 15
3	14	13, 15	12, 17	11, 19
4	16	15, 18	14, 20	13, 23
5	19	17, 20	16, 23	15, 26
6	21	20, 23	18, 26	16, 29
7	23	21, 25	19, 29	18, 32
8	25	23, 27	21, 31	20, 35
9	27	25, 29	23, 34	21, 38
10	29	27, 32	24, 36	23, 41
20	46	42, 50	38, 58	35, 65
30	60	55, 66	50, 77	47, 86
40	74	68, 82	61, 94	57, 105
50	87	80, 96	72, 111	67, 124
60	100	91, 110	82, 127	77, 142
70	112	102, 124	93, 143	87, 160
80	124	113, 137	102, 158	96, 177
90	136	124, 151	112, 174	105, 194
100	148	135, 164	122, 189	114, 211

Table 4. $\alpha = 500$

D	N_D	L_D^H, U_D^H		
		$\varepsilon = 1\%$	$\varepsilon = 5\%$	$\varepsilon = 10\%$
1	22	20, 25	17, 30	15, 34
2	32	30, 35	27, 40	25, 45
3	40	37, 44	33, 50	31, 56
4	47	43, 51	39, 59	36, 66
5	53	48, 58	44, 67	41, 74
6	58	53, 64	48, 74	45, 82
7	63	58, 70	52, 80	49, 90
8	68	62, 75	56, 86	53, 97
9	73	67, 80	60, 92	56, 103
10	77	70, 85	64, 98	59, 109
20	113	103, 125	93, 144	87, 161
30	143	130, 158	118, 182	110, 204
40	169	154, 187	139, 215	130, 241
50	193	176, 213	158, 246	148, 275
60	215	196, 238	177, 274	165, 307
70	236	215, 261	194, 301	182, 337
80	256	233, 284	211, 327	197, 366
90	275	251, 305	227, 352	212, 394
100	294	268, 326	242, 376	226, 421

general problem. Several researchers have tried to attack this problem but still unsuccessfully.

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Biography

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