Scale-invariant quenched disorder and its stability criterion at random critical points

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The critical properties of systems with quenched bond disorder are determined from a fixed distribution, under renormalization group, of the random bonds. Full fixed distributions with all moments are obtained numerically by histograms and, to a good approximation, in terms of Γ distributions. For such systems, the specific-heat exponent α does not equal the crossover exponent ϕ at random criticality. We derive a new relation between α and ϕ, which invokes characteristics of the fixed distribution. The difference between α and ϕ is noted for n-vector models in $4-\epsilon$ dimensions and for Potts models on hierarchical lattices solved exactly. In general, stable random critical behavior with positive α appears to be possible. We develop a general treatment of quenched disorder and illustrate it by calculating specific-heat curves. It is suggested that the critical exponents of the three- and four-state random-bond Potts models in two dimensions are $\nu \simeq 1.06$ and 1.19.

Properties of a system near its critical point are dominated by the statistical mechanics of fluctuations occurring at all length scales, and therefore can be obtained by analysis of a Hamiltonian which is invariant under rescaling transformation.[1,2] This of course is the fixed point of a renormalization-group transformation.[2] If quenched disorder is present and affects long-range fluctuations, the new criticality is expected to be determined by a scale-invariant form of the quenched disorder. This is a fixed distribution. In this paper, a renormalization-group treatment of systems with quenched disorder is developed. From fixed distributions and from distributions evolving under renormalization-group iterations, properties at and away from criticality are deduced.

An important advance in the study of criticality under quenched bond disorder was achieved with a physical argument due to Harris.[3] The self-consistency condition for a single well-defined critical temperature throughout the system was found to be a negative specific-heat exponent, $\alpha<0$. Actually, this compact form is the accident of a hyperscaling relation between critical exponents, as the argument yields $2-d\nu<0$, where ν is the correlation-length exponent and d is the spatial dimensionality. The latter expression was in fact identified as the crossover exponent ϕ at the pure-system criticality, with respect to small quenched disorder. This renormalization-group result was obtained by Aharony,[4] who studied the decoupling fixed point of a "replicated" system. It was later reproduced by Chalupa,[5] and by Kinzel and Domany,[6] without recourse to the replica trick in dealing with quenched disorder, but with a heuristic renormalization-group argument. Accordingly, a system with pure $\alpha<0$ exhibits pure-system criticality when perturbed by quenched bond disorder. Conversely, pure-system criticality is asymptotically lost under such perturbation, if the pure α is positive.

Yet another simple and heuristic derivation of this criterion is as follows. To a pure system at criticality introduce small quenched local fluctuations of average magnitude V and zero mean. In a Kadanoff block[1] of length b along each dimension, the summed fluctuation is $b^{d/2}V$. We heuristically hypothesize that this has the same effect as when the summed fluctuation is equally distributed among all the sites inside the block, namely $b^{-d/2}V$ per site. A renormalization-group transformation[2] is carried out and the block is replaced by a single site of the renormalized system. The quenched fluctuation associated with the renormalized site is

$$V'=b^{y_V}b^{-d/2}V\,, \tag{1}$$

where y_V is the eigenvalue exponent of the pure system corresponding to the variable which is being perturbed by the quenched fluctuations. Thus, the latter are relevant or irrelevant depending on whether $y_V-d/2$ is positive or negative. When V is a bond strength, the criterion is equivalent to the sign of the specific-heat exponent α. When V is a magnetic field, the criterion is equivalent to the sign of the susceptibility exponent γ.[7] The remainder of the article will be in terms of random bonds, although our approach is equally applicable to random fields.

In a general model of quenched bond disorder, the local bond strengths K_r are frozen in a random configuration $\{K\}$. The ensemble of such configurations is described by a probability distribution $P(\{K\})$. The quantitative statistical mechanics is performed by monitoring the effect of a renormalization-group transformation on this probability distribution,[8]

$$P'(\{K'\})=\int\left[\prod_r^N dK_r\right]\left[\prod_{r'}^{N'}\delta(K'_{r'}-R_{r'}(\{K\}))\right]P(\{K\})\,, \tag{2}$$

where primes refer to the renormalized system, $\{R(\{K\})\}$ are the set of recursion relations for a specific bond distribution $\{K\}$, and the index r distinguishes the N members of the set $\{K\}$. The present work utilizes the approach embodied in Eq. (2), including the cases of fully developed quenched randomness (as opposed to infinitesimal perturbation about a pure system) and the cases where new types of critical behavior are caused by the randomness. The latter situation is reflected by a fixed distribution,

$P^*(\{K\})$, of finite width and shape, which is invariant under the renormalization-group transformation of Eq. (2), and onto which all systems with the new criticality collapse under Eq. (2). One specific new result[6] is that the exponent equality $\phi=\alpha$ does not apply at random criticality. This is generally true, even at random criticality infinitesimally close to pure-system behavior. The direction of the ensuing inequality is not generally determined, so that predominant critical behavior ($\phi<0$) with quenched impurities and positive α is, in principle, possible. These results will be derived analytically. We have also numerically pursued the transformation of Eq. (2) and determined fixed distributions. From the rescaling behavior of small functional deviations from these fixed distributions, critical exponents have been calculated. The difficulty introduced by the unstable (temperaturelike) deviation has been surmounted. These fixed distributions are found to be well approximated by Γ distributions. These calculations and results are illustrated with q-state Potts models,[9] with α conveniently variable[10] as a function of q, in a position-space renormalization-group context. We also show that momentum-space renormalization-group on n-vector spins in $4-\epsilon$ dimensions support these findings.

Our present treatment is not without simplifying assumptions. We restrict ourselves to probability distributions which factorize into single-bond distributions,

$$P(\{K\})=\prod_r^N p(K_r) . \tag{3}$$

This restriction assumes that the transformation does not generate appreciable quenched correlations between the probabilities of distinct renormalized bonds. If, on the other hand, appreciable correlations were induced in the region of the renormalization-group flows, the global probability distribution would then be factorized into c-bond distributions, $\prod_r p(K_r,K_{r+1},\dots,K_{r+c})$, where c is the number of bonds inside a quench-correlated region.[11] Thus, the analysis below would proceed, with the scalar K_r replaced by the vector $\vec{K}_r=(K_r,K_{r+1},\dots,K_{r+c})$, provided the generated quenched correlations are not long ranged.[12] This caveat is analogous to the condition that no appreciable long-range interactions be generated in the renormalization-group treatment of pure systems.[13] Presently adopting Eq. (3), the transformation (2) reduces to

$$p'(K'_{r'})=\int \left[\prod_r^m dK_r p(K_r)\right]\delta(K'_{r'}-R_{r'}(\{K\}_m)) . \tag{4}$$

Note that in the thermodynamic limit, for each renormalized bond $K'_{r'}$, there are $m=b^d$ original bonds K_r, where b and b^d are, respectively, the length and volume rescaling factors. Another simplification, already incorporated into Eq. (4), results from the assumption that each $K'_{r'}$ depends appreciably only on the b^d original bonds K_r which are inside the rescaling volume of this given $K'_{r'}$. Actually, the factorization and $m=b^d$ assumptions are similarly argued, and their breakdown can be cured by the same development mentioned above. These simplifications are also used in the nonreplica renormalization-group arguments for the Harris criterion at pure criticality,[5,6] and are exact in the Migdal-Kadanoff procedure,[14,15] on hierarchical lattices,[10,16] and in the limit of large-rescaling position-space transformations.[17]

Consider a fixed distribution, $p(K)=p^*(K)$, with mean

$$\mu_1^*=\int dK\, p^*(K)K , \tag{5a}$$

and variance

$$\mu_2^*=\int dK\, p^*(K)(K-\mu_1^*)^2 . \tag{5b}$$

To analyze critical behavior, a small deviation from $p^*(K)$ is considered: Each bond of the system is subjected to a small, random deviation ΔK according to the distribution $\delta p(\Delta K)$, symmetric about the mean δ_1 and with variance δ_2 such that $\delta_1,\delta_2 \ll \mu_2^*$. The overall bond-strength distribution is the convolution

$$p(K)=\int dx\, p^*(K-x)\delta p(x) \simeq p^*(K)-\delta_1\frac{dp^*(K)}{dK}+\frac{\delta_2}{2}\frac{d^2p^*(K)}{dK^2} , \tag{6}$$

to leading order in $\delta_{1,2}$. The deviations in the moments of $p(K)$ are

$$\delta\mu_1\equiv\mu_1-\mu_1^*=\delta_1 , \quad \delta\mu_2\equiv\mu_2-\mu_2^*=\delta_2 , \tag{7}$$

where $p^*(K\to\pm\infty)=0$ is used. The effect of rescaling, via Eq. (2), is to leading order

$$\begin{bmatrix}\delta\mu'_1\\ \delta\mu'_2\end{bmatrix}=\begin{bmatrix}\langle\partial_1 R\rangle & \langle\partial_2 R\rangle\\ (\langle\partial_1 R^2\rangle-2\langle R\rangle\langle\partial_1 R\rangle) & (\langle\partial_2 R^2\rangle-2\langle R\rangle\langle\partial_2 R\rangle)\end{bmatrix}\begin{bmatrix}\delta\mu_1\\ \delta\mu_2\end{bmatrix} , \tag{8}$$

where

$$\partial_n\equiv\frac{1}{n}\sum_{i=1}^m\frac{\partial^n R(\{K\})}{\partial K_i^n} ,$$

$m=b^d$, and $\langle\cdots\rangle$ indicates averaging with respect to $p^*(K)$, e.g.,

$$\langle R\rangle=\int\left[\prod_r^m dK_r p^*(K_r)\right]R(\{K\}) . \tag{9}$$

An immediate check is to consider the special case of the fixed distribution $p^*(K)=\delta(K-K^*)$, namely, a pure system. In this case, $\langle R\rangle=K^*$ and $\langle\partial_1 R\rangle=b^{y_T}$. One eigenvalue of the recursion matrix in Eq. (8) is $\lambda_1=b^{y_1}$, $y_1=y_T$, with eigenvector purely in the nonrandom direction $\delta\mu_1$. The other eigenvalue is $\lambda_2=b^{y_2}$, $y_2=2y_T-d$, with eigenvector having components in both $\delta\mu_1$ and $\delta\mu_2$ directions.[18] Since $\alpha=2-d/y_1$ and $\phi=y_2/y_1$, the exponent equality $\phi=\alpha$ is recovered at the pure-system criticality. However, at random criticality, with fixed distribution $p^*(K)$ which is not a δ function, the eigenvalues are still easily obtained by diagonalizing the 2×2 matrix in Eq. (8), and $y_2\neq 2y_1-d$. Thus, the exponent equality is violated at random criticality. When $\mu_2^*\ll\mu_1^*$,

$$y_1\simeq y+\left[\partial_1(\partial_2 R)+\partial_2 R\left[2\sum_{i=1}^{m}\frac{\partial R}{\partial K_i}\frac{\partial(\partial_1 R)}{\partial K_i}-\partial_1^2 R\right]\frac{1}{b^y-1}\right]\frac{\mu_2^*}{b^y\ln b}\,,$$
$$y_2\simeq\left[4\sum_{i=1}^{m}\frac{\partial R}{\partial K_i}\frac{\partial(\partial_2 R)}{\partial K_i}-4\partial_2 R\left[\sum_{i=1}^{m}\frac{\partial R}{\partial K_i}\frac{\partial(\partial_1 R)}{\partial K_i}\right]\frac{1}{b^y-1}+\sum_{i,j=1}^{m}\left[\frac{\partial^2 R}{\partial K_i\partial K_j}\right]^2\right]\frac{\mu_2^*}{\ln b}\neq 2y_1-d\,, \quad (10)$$

where all derivatives are evaluated at the pure-system fixed point, and y is the pure-system thermal exponent, so that $\phi\neq\alpha$ at random criticality infinitesimally close to pure behavior.

We now demonstrate the numerical determination of a fixed distribution $p^*(K)$. A fixed distribution corresponding to a phase transition has an unstable direction of flow under renormalization, so that it is difficult to obtain by straightforward iteration of Eq. (4). Our approach is to narrow down initial conditions onto a phase boundary, flow under Eq. (4) to the neighborhood of $p^*(K)$, and use a Newton-Raphson algorithm to determine $p^*(K)$ for successively finer numerical representations. The range of bond-strength values is divided into M intervals, and the distribution is approximated by

$$p_M(K)=\sum_{i=1}^{M}p_i\Delta_i(K)\,, \quad (11)$$

where $\Delta_i(K)$ equal 1 if K falls into the ith interval, and is zero otherwise. The ith interval is centered at K_i and has width k_i. The successive numerical refinements are accomplished by increasing M, which is readily taken to a desired accuracy. The recursion relation of Eq. (4) can now be cast as projections between histogram probabilities,

$$p_i'=k_i^{-1}\sum_{l_1=1}^{M}\cdots\sum_{l_m=1}^{M}\Delta_i(R(K_{l_1},\ldots,K_{l_m}))\prod_{j=1}^{m}k_{l_j}p_{l_j}\,. \quad (12)$$

A cutoff K_u is simply implemented by counting all $R>K_u$ into the last histogram before K_u. Setting the fixed-point condition $p_i'=p_i\equiv p_i^*$ for all i and noting the normalization condition $\sum_i p_i k_i=1$ [which is of course conserved by Eq. (12)], leave $M-1$ coupled fixed-point equations to solve, which is where the Newton-Raphson algorithm is used on successive values of M. We tested this approach using the Migdal-Kadanoff[14,15] recursion relation for the q-state Potts models, with $b=d=2$, equivalent to the exact solution of a hierarchical model:[10,16]

$$R(K_1,K_2,K_3,K_4)=\ln[(e^{K_1+K_2+K_3+K_4}-1+q)/(e^{K_1+K_2}+e^{K_3+K_4}-2+q)]\,. \quad (13)$$

(It is to be stressed that the function R is an input in our approach.) For $q=18.75$, the pure-system criticality has specific-heat and crossover exponents equal to 0.333, and therefore is unstable to quenched impurities. This particular q value is chosen as our example, because it has pure α equal to that of $q=3$ on $d=2$ Bravais lattices.[19] The fixed distribution $p^*(K)$ which controls random criticality is shown in Fig. 1(a). It was also found that $p^*(K)$ is well approximated by a Γ distribution [Fig. 1(b)],

$$p^*(K)\simeq\Gamma(K)=6.07K^{1.79}e^{-2.29K}\,. \quad (14)$$

Substituting the general form of a Γ distribution into Eqs. (4) and (13), three of the four integrals can be performed analytically. A quasifixed distribution is located by numerically searching through the fourth integral.

The renormalization-group eigenvalue analysis can be done with the rescaling behavior either of the histogram probabiltiies, or of the moments. For small deviations from the fixed distribution,

$$p_i'-p_i^*=\sum_{j=1}^{M}T_{ij}(p_j-p_j^*)\,,$$
$$\mu_i'-\mu_i^*=\sum_j\widetilde{T}_{ij}(\mu_j-\mu_j^*)\,, \quad (15)$$

where

$$\mu_1=\sum_{j=1}^{M}k_jp_jK_j \quad\text{and}\quad \mu_{n>1}=\sum_{j=1}^{M}k_jp_j(K_j-\mu_1)^n\,.$$

The two recursion matrices T and $\widetilde{T}$, being related by a similarity transformation, have the same eigenvalues $\lambda_i=b^{y_i}$. Table I gives the eigenvalue exponents y_i at $p^*(K)$ of Fig. 1(a). Only one positive (relevant) exponent $y_1=0.94$ is seen, corresponding to the eigendirection out of the phase boundary. All other exponents are negative (irrelevant), corresponding to eigendirections within the phase-boundary hypersurface. Specifically,

$$y_2=-0.30\neq 2y_1-d=-0.12\,.$$

TABLE I. Leading eigenvalue exponents and the components of their right eigenvector along the leading moments $(\mu_n)^{1/n}$ of the bond probability distribution. Also given are the mean $\bar{K}^*=\mu_1^*$ and the width $\sigma^*=(\mu_2^*)^{1/2}$ of the fixed distributions. These results are obtained for the $q=18.75$ (pure $\alpha=0.333$) Potts model under the $b=d=2$ Migdal-Kadanoff recursion. The fixed distribution was successively approximated by M histograms of equal width spanning the interval $0<K<3.6$. The first eigenvector, corresponding to the temperature direction, has its largest component along the mean (the first moment) of the distribution. The other eigenvectors, corresponding to the irrelevant ($y_i<0$) exponents, lie almost entirely along the higher moments.

	Eigenvalue exponents			Eigenvector					
	$M=8$	16	32	$M=32$					
y_1	0.887	0.926	0.940	0.3753	0.0982	−0.0343	0.0577	−0.0292	−0.0033
y_2	−0.544	−0.367	−0.300	0.0598	0.2485	0.2271	0.2765	0.2695	0.2724
y_3	−1.841	−1.639	−1.518	0.0234	0.0691	0.2535	0.1675	0.2622	0.2250
y_4	−2.665	−2.715	−2.604	0.0098	0.0391	0.1317	0.1509	0.2036	0.2129
y_5	−3.188	−3.627	−3.510	0.0060	0.0225	0.0919	0.1023	0.1800	0.1795
y_6	−3.732	−4.119	−4.263	0.0038	0.0155	0.0632	0.0787	0.1402	0.1588
y_7	−7.632	−5.145	−5.002	0.0027	0.0106	0.0467	0.0584	0.1127	0.1304
y_8		−5.453	−5.484	0.0020	0.0084	0.0358	0.0481	0.0924	0.1133
$\bar{K}^*$	1.258	1.252	1.236	Exponents $M=32$					
σ^*	0.873	0.855	0.818		$\alpha=-0.128$			$\phi=-0.319$	

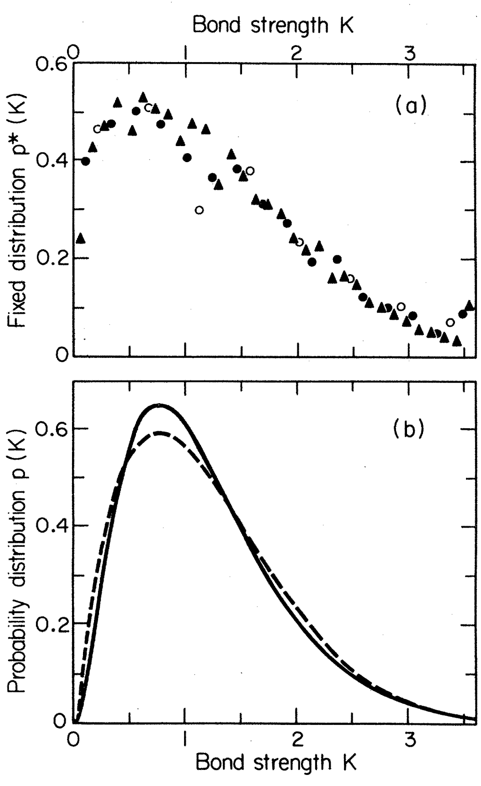

FIG. 1. (a) Fixed distribution of the $q=18.75$ (pure $\alpha=0.333$) random-bond Potts model under the $b=d=2$ Migdal-Kadanoff recursion. $M=8$ (○), 16(●), and 32(△) histograms of equal width are used in successively more refined evaluations. The points next to $K_u=3.6$ are somewhat enhanced due to our cutoff procedure at that value. (b) The distribution $6.07\ K^{1.79}\exp(-2.29K)$ which approximates a fixed distribution for $q=18.75$. Dashed line is its first renormalization-group iteration.

The specific-heat and crossover exponents have the respective values $\alpha=-0.13$ and $\phi=-0.32<\alpha$. Figure 1(a) and Table I show that our procedure is convergent, by comparison of results with $M=18$, 16, and 32.

Leading exponents at pure and corresponding random criticality are given in Table II for several other calculated cases. Pursuing the philosphy that q can be used as an adjustable parameter[10] under the Migdal-Kadanoff recursion, to match and extrapolate from known pure-system criticality on Bravais lattices, we matched pure $\alpha=\frac{1}{3}$ and

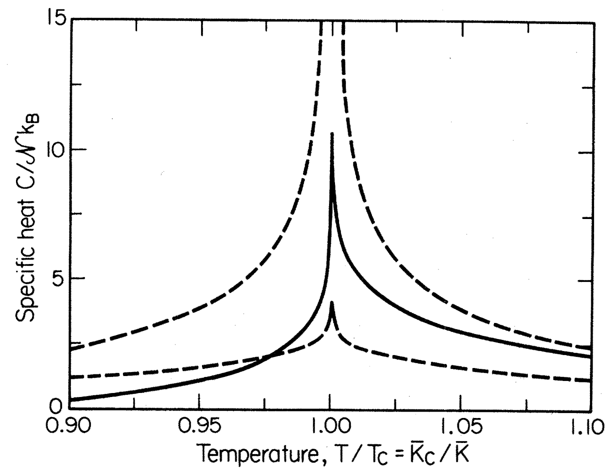

FIG. 2. Specific heat per site for q-state Potts models evaluated under the $b=d=2$ Migdal-Kadanoff recursion. The upper dashed curve shows the pure $q=18.75$ Potts model, which has the critical exponent $\alpha=0.333$. The full curve shows the random-bond $q=18.75$ Potts model, with $\alpha=-0.160$. The mean bond strength $\bar{K}=\mu_1$ was scanned by varying $\tilde{p}$ in the distribution $p(K)=\tilde{p}\delta(K-0.5625)+(1-\tilde{p})\delta(K-1.9125)$, so that the variance remained in the range $0.43<\mu_2<0.46$. $M=16$ histograms and an upper cutoff of $K_u=3.6$ were used in this calculation. Finally, the lower dashed curve shows the pure $q=4.77$ Potts model, which has the same critical exponent $\alpha=-0.160$ as the random $q=18.75$ model. Note the asymmetry which develops in the random case.

TABLE II. Random and pure criticality of q-state Potts models under the $b=d=2$ Migdal-Kadanoff recursion. The fixed distributions, with mean $\bar{K}^*=\mu_1^*$ and width $\sigma^*=(\mu_2^*)^{1/2}$, were obtained with $M=16$ histograms of equal width, using the upper cutoff values of $K_u=2.2$, 2.5, 3.6, 4.0, and 8.0, respectively. A better representation of $q=18.75$ random criticality is given in Table I with $M=32$.

	Random criticality						Pure criticality	
q	$\bar{K}^*$	σ^*	y_1	y_2	α	ϕ	y_T	$\alpha=\phi$
12	1.07	0.50	0.964	−0.302	−0.075	−0.313	1.114	0.205
14	1.12	0.59	0.957	−0.307	−0.090	−0.321	1.144	0.252
18.75	1.25	0.85	0.926	−0.367	−0.160	−0.396	1.200	0.333
25	1.36	0.99	0.909	−0.426	−0.200	−0.469	1.220	0.405
116	1.92	1.66	0.843	−0.520	−0.372	−0.617	1.500	0.667

$\frac{2}{3}$ with q(adjusted)$=18.75$ and 116. Thus, it is suggested that the corresponding random bond $\alpha=-0.13$ and -0.37 (or, equivalently, $\nu=y_1^{-1}=1.06$ and 1.19) apply to the 3- and 4-state Potts models on $d=2$ Bravais lattices. The latter result is close to, but outside the reported uncertainty ($\nu=1.0\pm0.07$) of a Monte Carlo simulation[20] of the random Baxter-Wu model,[21] which should be in the universality class of the $q=4$ Potts model. Random $d=2$ Potts models could be important[22,23] for the interpretation of critical behavior in adsorbed systems.[24]

In light of these new results, we have reexamined results of momentum-space renormalization-group calculations,[8,25] for n-vector spins in $d=4-\epsilon$ dimensions. It is indeed seen that, at random criticality,

$$\phi=\alpha+\frac{3n(5n+4)(n-4)\epsilon^2}{64(n-1)^2(5n-8)}+O(\epsilon^3)$$
$$=\alpha-(n_c-n)\epsilon^2/24+O((n_c-n)^2,\epsilon^3)\ , \qquad (16)$$

where $n_c=4-4\epsilon$ is the component number below which quenched impurities become relevant. The inequality $\phi<\alpha$ sets at second order in ϵ and was not noted in previous works.

In both examples above, Potts models under the Migdal-Kadanoff renormalization procedure, or on hierarchical lattices solved exactly, and n-vector spins in $4-\epsilon$ dimensions, $\phi<\alpha<0$ at stable random criticality. From our general formulas, it is in principle possible[6] to have these two exponents of opposite sign, $\phi<0<\alpha$, depending on the characteristics of the fixed distribution. However, we have not yet located an example of this.

Beyond these critical exponents, we have developed here a treatment for systems strongly affected by quenched randomness. Entire thermodynamic curves can be calculated, as examplified by the specific heat shown in Fig. 2. The contribution to the free energy per original bond from the nth renormalization is

$$\bar{f}^{(n)}=b^{-dn}\sum_{l_1=1}^{M}\cdots\sum_{l_m=1}^{M}G(K_{l_1},\ldots,K_{l_m})\times\prod_{j=1}^{m}k_{l_j}p_{l_j}^{(n-1)}\ , \qquad (17)$$

where $G(\{K\}_m)$ is the additive constant generated in the local rescaling, which, in Eq. (17), is averaged over the iterated bond distribution. For example,

$$G(K_1,K_2,K_3,K_4)=\ln(e^{K_1+K_2}+e^{K_3+K_4}-2+q) \qquad (18)$$

in the above Migdal-Kadanoff procedure. The total free energy per original bond, $\sum_n\bar{f}^{(n)}$, rapidly converges with the number of renormalizations n, as in other position-space treatments. Numerical differentiation yields the specific heat. The critical singularity is depressed in temperature compared with the pure system and appears to have developed asymmetry. We hope that this approach will be useful in other interesting problems, such as random tricriticality[23] and random fields.[26]

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