Version: 02/18/08

Buying Stock on Margin Can Reduce Retirement Risk

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Abstract: The typical person invests far too little in stocks when young. Since the young are liquidity constrained, the only way to invest more is to buy stocks with leverage. While leveraged purchases of stock increase short-term risk, it reduces long-term risk by letting individuals achieve better diversification across time. To reduce retirement savings risk, people should move closer to investing equal dollar amounts in stock each year of their working life. We derive a four-phase life-cycle strategy that improves temporal diversification of retirement investments. Using stock data going back to 1871, we show that buying stock on margin when young combined with more conservative investments when older stochastically dominates standard investment strategies—both traditional life-cycle investments and 100%-stock investments. The expected retirement wealth is 84% higher compared to life-cycle funds and 20% higher compared to 100% stock investments. Relative to traditional life-cycle investments, the expected gain from this improved asset allocation would allow workers to retire 9 years earlier or extend their standard of living during retirement for an additional 20 years. For a worker with CRRA=2, the increased retirement consumption raises lifetime utility by 7.3%. The potential gains are substantial.

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James Poterba and Robert Shiller provided helpful comments. Isra Bhatty, Jonathan Borowsky, and Heidee Stoller provided excellent research assistance. A preliminary investigation into the issues covered in this paper was published in *Forbes* (see Ayres & Nalebuff (2005)). The data and analysis underlying our estimates can be downloaded from www.law.yale.edu/ayres.

1. Introduction

The typical decision of how to invest retirement savings is fundamentally flawed. The standard advice is to hold stocks roughly in proportion to 110 minus one's age. Thus a twenty-year old might be 90-10 in stocks versus bonds, while a sixty year old would be 50-50. This advice has been automated by lifecycle funds from Fidelity, Vanguard, and others that each year shift the portfolio from stocks into bonds.[1] Our results demonstrate that the early asset allocation is far too conservative.

We find that people should be holding much more stock when young. In fact, their allocation should be more than 100% in stocks. In their early working years, people should invest on a leveraged basis in a diversified portfolio of stocks. Over time, they should decrease their leverage and ultimately become unleveraged as they come closer to retirement. The lifetime impact of the misallocation is large. The expected gain from this improved asset allocation relative to traditional life-cycle investments would allow people to retire approximately nine years earlier or to retire at the same age (65) and yet maintain their standard of living through age 106 rather than age 85.

The insight behind our prescription comes from the central lesson from finance: the value of diversification. Investors use mutual funds to diversify over stocks and over geographies. What is missing is diversification over time. The problem for most investors is that they have too much invested late in their life and not enough early on.

The recommendation from the Samuelson (1969) and Merton (1969, 1971) life-cycle investment models is to invest a constant fraction of wealth in stocks. The mistake in translating this theory into practice is that young people invest only a fraction of their *current* savings, not their discounted lifetime savings. For someone in their 30's, investing even 100% of current savings is still likely to be less than 10% of their lifetime

[1] Both the Fidelity Freedom Funds and Vanguard's Target retirement funds start with 90% in stocks and 10% in bonds and gradually move to a 50-50% allocation at retirement. The initial rampdown is slower than linear; for example, Vanguard stays at 90% through age 40. See
http://personal.fidelity.com/products/funds/content/DesignYourPortfolio/freedomfunds.shtml.cvsr and
https://flagship.vanguard.com/VGApp/hnw/content/Funds/FundsVanguardFundsTargetOverviewJSP.jsp

savings or less than 1/6th of what the person should be holding in equities if their risk aversion led them to invest 60% of their lifetime savings.

In the Samuelson framework, all of a person's wealth for both consumption and saving was assumed to come at the beginning of the person's life. Of course that isn't the situation for a typical worker who starts with almost no savings. Thus, the advice to invest 60% of the present value of future savings in stocks would imply an investment well more than what would be currently available.

This leads to our simple advice: *buy stocks using leverage when young*. One way to have more invested in the market when young is to borrow to buy stocks. This is the typical pattern with real estate where the young take out a mortgage and thereby buy a house on margin. We propose that people follow a similar model for equities.

Practically speaking, people have limited ability to borrow against their future earnings. But they can buy stock on margin or gain leverage by buying stock derivatives. If a young investor with $10,000 in savings and a lifetime wealth of $100,000 were to buy stock on 2:1 margin, the resulting $20,000 investment would still leave her well short of the desired $60,000 in equities. Buying stocks on 3:1 margin would get her half way there. Both strategies are better than limiting investments to 90% or even 100% stocks.

Another approach to gain leverage is to buy index option contracts that are well in the money. For example, a two-year call option with a strike price of 50 on an index at 100 will cost something close to 50. Thus for $50, the investor can buy exposure to $100 of the index return. We show below that the implied cost of such 2:1 leverage is quite low (about 50 basis points above the yield on a one-year treasury note), which makes the strategy practical in current markets.

We recognize that our recommendation to begin with leverage positions goes against conventional advice. And yet, our recommendation flows directly from the basic Samuelson and Merton lifecycle savings model. It is also supported by the data. We will show that following this advice leads to higher returns with lower risks. This is true both for historical data and for a variety of Monte Carlo simulations.

We derive a four-phase allocation strategy with decreasing amounts of leverage in each phase. Like Samuelson and Merton, the core investment strategy in each phase is to invest a constant percentage of the present value of savings in stock, where the percentage is a declining function of risk-aversion. Because of the cost of borrowing on margin, the investment goal during the initial leveraged phases is lower than during the later unleveraged phases.

The desirability of this four-phase strategy relies on the existence of an equity premium. Leveraging only makes sense if the expected return on stock is greater than the implicit margin rate. In our data (going back to 1871), we find that equities returned 9% in real terms, while the real cost of margin was 5%. This 4% premium was the source of the increased returns of our leveraged life-cycle strategy. As Barberis (2000) observes, this equity premium is based on a relatively limited data and just one sample path; thus investors should not count on the equity premium persisting at historical levels. Shiller (2005a) goes further to suggest that the U.S. equity performance is unlikely to be repeated.[2] In our robustness section, we show that even with the equity premium reduced by half (or a higher margin rate) there is still a gain from more leverage for the young.

Our focus is on investment allocation during working years. We do not consider how the portfolio should be invested during the retirement phase—although results from Fontaine (2005) suggest that standard advice may be too conservative here as well.[3] Nor do we take on the difficult and interesting question of how much people should optimally save over the course of their lives. Instead, we focus on the allocation between stocks and bonds taking the savings rate as exogenously given. We show that for a typical vector of

[2] The high equity premium may also be an artifact of survivorship bias (see Brown, Goetzmann and Ross, 1995).

[3] This asset allocation during retirement can be avoided through the purchase of annuities, which also solves the problem of an uncertain lifetime.

saving contributions, our proposed investment strategy first-order stochastically dominates the returns of traditional investment strategies.

The assumption of exogenous savings is reasonable. Many people save money for retirement via automatic payroll deduction (Poterba and Samwick 2001). There are tax advantages to putting aside money in a relatively illiquid 401(k) plan and these contributions are often matched by the employer. Even young workers who are constrained in terms of consumption might still choose to put something away toward retirement. Whether savings are optimal or not, we argue that any retirement savings that do occur should initially be invested on a leveraged basis so that more than 100% of the net portfolio value is in equities.

With the shift away from defined benefit to defined contribution pensions, much of early savings comes from tax-advantaged and employer-matched 401(k) plans. Thus our advice is especially relevant for the allocation of stocks inside a 401(k) plan. Unfortunately, current regulations effectively prevent people from following our advice with regard to their 401(k) investments. The reason is that an employer could lose its safe-harbor immunity for losses if any one of its plan offerings is later found by a court to not be a prudent investment. Allowing employees to buy stocks on margin is not yet considered prudent, although we hope this analysis will help change that perspective.[4]

Of course, borrowing on margin creates a risk that the savings will be entirely lost. That risk is related to the extent of leverage. If portfolios were leveraged 20 to 1, as we do with real estate, this risk would be significant. We propose a maximum leverage of 2:1. It is worth emphasizing that we are only proposing this amount of leverage at an early stage of life. Thus, investors only face the risk of wiping out their current investments when they are still young and will have a chance to rebuild. Present savings might be extinguished, but the present value of future savings will never be. Our simulations

[4] However, it is possible to create the equivalent to leveraged positions in self-directed IRAs and Keogh plans by investing in options on stock indexes; see http://www.cboe.com/institutional/irakeogh.aspx.

account for this possibility and even so, we find that the *minimum* return under the strategies with initially leveraged positions would be substantially higher compared to the minimum under traditional investment strategies.

Our core analysis ignores the impact of human capital or housing investments on the optimal retirement investment. As emphasized by Viceira (2001) and Campbell and Viceira (2002), many people, especially the self employed, are already heavily invested in the market via human capital. To the extent that human capital is correlated with the market, then the person might be fully invested in equities.[5] The degree of correlation is an empirical question that varies by profession. In academia, for example, senior faculty salary increases generally run slightly above inflation.[6] Future salary is much less volatile than the stock market. Thus, even taking human capital exposure to stock market risk into account, assistant professors and many others should still invest on margin when young.

The point of this paper is to overturn the standard orthodoxy that counsels against buying stock on margin. Many people (including ourselves) misinvest their retirement portfolio when young (Poterba 2005). The cost of this mistake is not small. Our estimates suggest that if people had followed this advice historically they would have retired with portfolios worth 20% more on average compared to all stock and 84% more when compared to the lifecycle strategy. These gains could be socially significant. Poterba, Rauh, Venti and Wise (henceforth PRVW (2005a) report that in 2000 lifecycle funds held $5.5 billion, and that their assets had grown to $47.1 billion by 2005. Hewitt Associates estimates that 38% of all 401(k) plans offer lifecycle funds (Marquez 2005). Of course, if everyone were to follow our advice, there might be some general equilibrium effects that could lead to lower stock returns. So far, this is not an issue.

The increased returns also have less risk. Based on historical data, we find that the

[5] At the extreme, Benzoni, Collin-Dufresne, and Goldstein (2004) show that a risk-averse (γ=5) young worker may actually want to *short* equities. The reason is the high cointegration of the labor and equity markets. Because wages depend on profits, the young risk-averse worker is already overinvested in the market through her human capital.

[6] Survey data of American Association of University Professors, reported in http://chronicle.com/stats/aaup.

margin purchases lead to a first-order stochastic dominant set of returns. For all risk preferences, the results are better. This suggests a simple rule that will lead to better outcomes: whatever savings young people have, they should leverage them up.

2. Connection to the Literature

The theory approach to lifecycle portfolio allocation begins with Samuelson (1969) and Merton (1969). They demonstrated that the allocation between equities and bonds should be constant over the life cycle. The allocation depends only on the degree of risk aversion and the return on equities, not age.

Samuelson was responding to the view that young investors should take more risks because they had more years with which to gamble. This was the "intuition" that supported investment advice such as the "110 − Age" rule. It is interesting that in spite of nearly forty years of contraindication from theory, the rule is still recommended practice.[7]

It is easy to become confused about whether an investment when young or old is riskier. An investment when young gets amplified by the returns of all subsequent years. An investment when old multiplies all of the previous returns. This vantage suggests that the two investment periods contribute the same amount of risk towards consumption in retirement.

To see this intuition, consider the two-period allocation problem where z_i is the return in period i and λ is the allocation of assets to equities. The investor chooses λ_1 and λ_2 to maximize:

[7] For example, Malkiel (2003) proposes a portfolio that is starts at 75% equities (including real estate), ramps down to 65% in the late 30s/early 40s, reduces to 57.5% exposure in the mid 50s, and falls to 40% at retirement. This is close to a 110 − Age rule. The Vanguard and Fidelity funds go from 90% at age 20 down to 50% at age 65, but they fall more slowly at first making them closer to 120 − age than 110 − age. While the Samuelson result assumes a constant relative risk aversion, it is hard to imagine that a "120 − Age" rule would arise due to a different utility function.

$$EU[W * (\lambda_1 z_1 + (1-\lambda_1)(1+r)) * (\lambda_2 z_2 + (1-\lambda_2)(1+r))].$$

Imagine, counterfactually, that the investor must make both allocation decisions prior to observing the returns.[8] (In practice, the person observes the first-period returns before making the second-period allocation.) Note the symmetry of the problem. The results of the first period are amplified by what happens in the second period. Thus if we expect that the second-period returns will be 10%, then it is as if the person is taking a 10% bigger gamble in the initial period. At the same time, the investor expects that wealth will be bigger in the second period, also by 10%. Thus the second-period investment is made on a larger wealth base. The investment decisions are symmetric. The investment in each period is amplified by the returns in all of the other periods.

The fact that investors can observe the results of previous investments allows some additional flexibility. However, in the case of constant relative risk aversion, there is no advantage from this extra information. The investor would choose the same allocation for all income levels and thus can make the decision without knowing the initial returns.

Moving outside the world of constant relative risk aversion offers a motivation for changing the equity allocation over time. The later period allocations can respond to changes in wealth. The early allocation might then respond to the fact that later allocations can adjust. This flexibility increases the attractiveness of investing, but whether it increases the marginal attractiveness when young is less clear.

A separate recommendation from the Samuelson model is that investments should be made as a fraction of lifetime wealth. In contrast, the lifecycle funds base investments on current savings, not on lifetime wealth. This is the most significant departure of practice from theory. For young workers, lifetime wealth is likely to be a large multiple of current

[8] While people are able to observe first-period returns prior to making the second-period allocation, they may not take advantage of this flexibility in practice. Employees in a 401(k) plan simply allocate their savings to 80% stocks and 20% bonds, for example, and then don't adjust the allocation based on market performance, except perhaps in the event of a crash or a bubble.

savings. Thus the only way to follow the Samuelson prescription is to invest using leverage.

In Samuelson, this issue is almost hidden since wealth is given exogenously up front. There is a large literature that considers how to translate future earnings into the initial wealth and the impact that has on current investment. See Bodie, Merton, and Samuelson (1992); Heaton and Lucas (1997); Viceira (2001); and Campbell and Viceira (2002); Benzoni, Collin-Dufresne, and Goldstein (2004); and Lynch and Tan (2004).[9] To the extent that human capital is correlated with equity returns, young workers might already be heavily invested in the equity markets. This also suggests that lifecycle funds should be different by profession, reflecting the different indirect exposure to equities via human capital.

To evaluate an allocation rule, we can look at its historical performance along with the results from Monte Carlo simulation. Poterba, Rauh, Venti, and Wise (2005a, 2005b) examine the performance of different portfolio allocation strategies over the lifecycle. Their basic finding is that maintaining a constant percentage in equities leads to similar retirement wealth compared to typical lifecycle strategies, holding the average equity allocation constant across strategies. In the empirical section, we compare our results to the equivalent constant percent strategy. Unlike PRVW we find that the leveraged investment strategy leads to substantially lower risk than the equivalent constant-equity percentage strategy. This is in accord with our intuition. The constant equity percentage (combined with exogenous savings) leads to an investment portfolio that grows something like $100, $200, $300, and more to the extent stock returns are positive. Our leveraged portfolio brings the investor closer to $200, $200, $200 and thus reduces overall risk.

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[9] In our model, we assume that retirement savings are exogenous and thus the only question is what discount rate to use, the margin rate or the bond rate. In the appendix, we show that the solution makes use of a fixed-point argument. Consider how much the person would want to invest when using the lower rate. If the person has that much to invest without leverage, then the lower interest rate is the right choice. Otherwise, this ends up being a target for when the investor has saved enough to reach this point without leverage.

The puzzle is why the traditional lifecycle strategies don't outperform the constant equity percentage. The answer is that the traditional lifecycle portfolios don't really change their allocation. Although they nominally move from 88% to 30% in the PRVW sample, since invested assets are so low during the early phase, the weighted average of 53% is much like the allocation during years 50 to 60 when the bulk of savings are made.[10] In contrast, our phased strategy starts at 200%, holds there for ten years (see Table 5), and then falls to 50%. Our strategy has a range of variation that cannot be replicated with a constant percentage. The equity allocation is designed to counterbalance the size of the savings, and this leads to a more even and thus less risky lifetime portfolio.

Shiller (2005a) considers a conservative life-cycle strategy, such as might be used for private social security accounts. The allocation to equities starts at 85% and ramps down to 15% at retirement age. This is much less exposure to equities than Vanguard and Fidelity lifecycle funds, which only fall to 50% equities at retirement. Shiller finds that investing 100% of current savings in stock throughout working life produces higher expected payoffs and even higher minimum payoffs than his conservative life-cycle strategy.

The prior literature establishes the equivalence of life-cycle to age-invariant asset allocation and the dominance of 100% allocations over a conservative life-cycle fund. Our contribution is to show that going beyond 100% equities further improves expected utility and that the gain is substantial: a 50% increased in expected retirement wealth compared to the 100%-equity strategy and a 123% increase compared to the typical Vanguard or Fidelity life-cycle fund.

Others have recognized the potential value of leverage. Viceira (2001) considers the investment allocation in a model where consumption and investment are both optimally chosen. His approach is based on finding a steady-state allocation. Thus a "young"

[10] In PRVW Table 1, they show that the average equity allocation falls to 30% upon retirement. In contrast, Fidelity and Vanguard are both at 50% at retirement date. It is possible that other funds are more conservative than Fidelity and Vanguard or that the idealized allocation of lifecycle funds has become more aggressive over time.

worker is one who has a small (but constant) chance of retiring each period. The allocation for older workers is the steady-state solution where the retirement probability is increased. The steady-state solution avoids the issue of workers having to build up savings from zero (which is the focus of our results). In Viceira's framework, the margin rate equals the bond rate. In a calibrated example where wages and equities are uncorrelated, he finds that "young" workers with low risk aversion (Constant Relative Risk Aversion (CRRA) = 2) will want to invest 292% of their wealth in equities. This falls to 200% when the worker only has an expected 22 years left in the workforce or if risk aversion were to rise to a little below 3.[11]

Closest to our work is Willen and Kubler (2006), who quantify the potential gain from investing retirement savings on a leveraged basis. Using similar parameters, they find that leveraging investments only leads to a 1.2% gain in utility relative to investing 100% of current assets in stock.[12] While the magnitude of their findings look quite different, the results are not as divergent as it might first appear. Willen and Kubler look at the present discounted value of lifetime consumption. For comparison, our expected 20% gain in the certainty equivalent of retirement wealth translates into a 3.3% gain in lifetime utility. The improvement is smaller because the gain is only during the years of retirement and the gains are delayed until the future, which is discounted.[13]

Whether a 3% gain in lifetime wealth is big or small depends on your perspective. The increased retirement wealth could be used to retire approximately 4.5 years earlier than a 100% stock investor. If retirement age is held constant, this expected gain in retirement wealth would allow people to maintain their standard of living for an additional 11 years of retirement or to age 96 (rather than 85).

[11] When the correlation between wages and equities rises to 25 percent, the young worker's allocation to equities falls by about 13%.

[12] This is with a 2:1 maximum leverage on margin accounts, and a 4% equity premium for stocks over the margin rate. See Willen and Kubler (2006, Table 8).

[13] Our 3.3% gain is still more than twice the estimates of Willen and Keebler. This difference is due to different modeling assumptions. Willen and Keebler emphasize the value of smoothing lifetime consumption. The high cost of borrowing against future income for consumption (10% in their model) means that most people consume too little when young. As a result, their investors do not begin to save for retirement until their early 50s, and this reduced period of investing substantially shrinks the gains from leverage.

Willen and Keebler also provide an answer to the equity participation puzzle. Given the large historical premium on equities, it would appear that people should hold significantly more equities. Their answer is that due to the high cost of unsecured borrowing to finance consumption, people would do better to consume more rather than save when young. [See also Constantinides, Donaldson, and Mehra (2002)] Our results suggest a different equity participation puzzle. To the extent that people aged 20 to 50 are saving for retirement in 401(k) plans and elsewhere, why aren't those savings all in equities and even more so, why aren't they leveraged on a 2:1 basis?

3. Investment Rule

Our four-phase investment strategy is an extension of the Samuelson (1969) and Merton (1969) result (henceforth SM) to take into account margin limits caused by the fact that investors do not start with all of their wealth upfront. As in SM, we assume that the investor's utility period function has constant relative risk aversion, U(x) = $\frac{x^{1-\gamma}}{1-\gamma}$ (where $\gamma > 0$ so that the individual is risk averse).[14] With these preferences and all wealth provided upfront, the optimal portfolio choice is independent of wealth. In addition, the optimal allocation can be calculated without knowing the consumption rule, assuming only that consumption is chosen optimally (or independently of retirement savings).

We recognize that most investors do not have all of their wealth upfront and thus may be liquidity constrained when young. For simplicity, we assume that future income is nonstochastic and that unleveraged equity investment is limited by liquid savings. This leads us to consider leverage and the relevant opportunity cost of buying equities. When investors are using leverage, the relevant forgone interest is the margin rate (as the investor could have paid down the debt); when investors invest without leverage, then the relevant foregone interest is the bond rate. Initially, we assume that these two rates are the same, and then extend the investment rule to the case where the margin rate is higher

[14] Note that for γ=1, the utility is defined as U(x) = ln(x).

than the bond rate.

As in SM, we consider a two-asset world where the risky asset can be thought of as stocks and the safe asset as bonds. The extension to include investing on margin is straightforward. We consider two interest rates, r_{m_i}, the real margin rate in period i, and the risk-free real rate, $r_{f_i} \le r_{m_i}$. For simplicity, we assume that the distribution of real stock and bond returns are i.i.d. over time and henceforth drop the *i* subscript. Associated with each interest rate is a target allocation rate, $\lambda(r_m)$ and $\lambda(r_f)$, respectively.

The investor's liquid savings are represented by S, and the person's PDV of future saving contributions is represented by W. The margin collateral rule requires that the investor put up $m of collateral for each dollar of equity. Thus the person with S of liquid assets is limited to buying S/m dollars of equities.

Initially, we assume that S=0. The investor starts out with no savings. Savings are built up from the 4% of income that is allocated to savings each period. Thus, initially, the investor will be constrained by the margin rule. The person will invest the maximum possible, S/m.

Over time, the investor will build up savings so that more of the person's wealth is liquid. At some point, the person will be able to reach the desired level allocation of wealth into equities. This is first done from a leveraged position and then done with diminished leverage as liquid assets continue to grow.

For example, under CRRA=2 and the historical returns, the optimal single period allocation is 88% to equities and 12% to bonds. Thus the investor works to build up to the point where 88% of S+W, his combined liquid savings plus the present value of future earnings, is invested in equities. This will be possible once 0.88*(S+W) < S/m.

This investment strategy is the translation of Samuelson and Merton, but it is no longer optimal in our framework. The reason is that the utility function is no longer multiplicative in wealth. Specifically, the margin constraint is not multiplicative in S+W. If the person's total wealth is doubled, but the liquid assets remain constant, then the person will not be able to double her investment in equities. Another way of seeing this is that if the stock return is very negative, the person may end up liquidity constrained in the next period. Thus the investment choice tomorrow is no longer independent of the decision made today.

To see this mathematically, consider the two-period problem (or the investment decision made for the last period before retirement):

$$J_1(W) = \max_c \frac{C^{1-\gamma}}{1-\gamma} +$$

i) $$(1+\rho)^{-1} * \frac{[S - C + \frac{w_2}{1+r_f}]^{1-\gamma}}{1-\gamma} * E[\frac{\lambda_b}{m} * z + (1 - \frac{\lambda_b}{m}) * (1 + r_f)]^{1-\gamma}$$

if $\lambda_b(S - C + \frac{w_2}{1+r_f}) < S - C$

ii) $$(1+\rho)^{-1} * \frac{1}{1-\gamma} * E[\frac{S-C}{m} * z + (S-C)(1 - \frac{1}{m})(1 + r_f) + w_2]^{1-\gamma}$$

if $\lambda_b(S - C + \frac{w_2}{1+r_f}) \geq S - C$

The choice of allocation to stocks ($\frac{\lambda_b}{m}$) depends on whether the person is liquidity constrained or not.

When there are two interest rates, one for lending (r_f) and one for borrowing on margin (r_m), our investment rule becomes a 4-phase path. Initially, the investor would like to be at $\lambda(r_m)$, but is unable to reach this allocation due to limits on the maximum leverage ratio. Thus the investor employs maximum leverage until $\lambda(r_m)$ is achieved (phase 1). The investor then deleverages her position while maintaining the $\lambda(r_m)$ allocation (phase 2). Once fully deleveraged, the new target is $\lambda(r_f)$. The investor allocates 100% of her available wealth in equities until this target is reached (phase 3). Finally (phase 4), the investor maintains the $\lambda(r_f)$ allocation, adjusting the portfolio based on changes in wealth.

In sum, what we will call the "two-target" investment strategy consists of four phases:

In phase 1: $\lambda < \lambda(r_m)$. All liquid wealth is invested at maximum leverage.
In phase 2: $\lambda=\lambda(r_m)$. The investor deleverages until $\lambda=\lambda(r_m)$ is achieved without leverage.
In phase 3: $\lambda(r_m) < \lambda < \lambda(r_f)$. The investor puts all liquid wealth into equities.
In phase 4: $\lambda = \lambda(r_f)$. The investor maintains the optimal SM allocation.

The discount rate determines both the current value of wealth (S+W) and the leverage target. The product of these two variables in turn determines the dollar amount to invest in equities, which determines whether the investor is liquidity constrained or not.

This 4-phase strategy has the advantage that it is characterized by just two percentage targets, $\lambda(r_m)$ and $\lambda(r_f)$. Furthermore, a person can get started on the optimal path even without knowing the initial target. A young investor who starts with little liquid assets will take several years to reach the first target, even when investing all liquid assets fully leveraged. In our simulations, we find that a person who saves 4% of her income remains fully leveraged until sometime between age 28 and 38 (95% confidence interval, see Table 5 below). Thus she can start down the optimal path even without knowing the final destination.

In our simulations, we explore the consequences of applying different parameters for each of these goals. The goals will vary with changes in the real interest rate, the margin premium, and the equity premium. In our simulations, we hold these parameters constant over the investor's life.

The level of the margin rate relative to the risk-free rate and the expected stock return has a large impact on the optimal investment strategy. If the margin rate equals the risk-free rate, i.e., if investors could borrow at the risk-free rate, $\lambda(r_m) = \lambda(r_f)$ and the third phase vanishes.[15] Investors maintain a constant SM percentage of wealth in stocks as soon as $\lambda(r_m)$ is reached. This single-target, three-phase strategy is relevant because, as an empirical matter, current margin rates are close to the risk-free rates and thus the two targets are also close. Thus we find that even the simpler single target, three-phase strategy performs almost as well as the four-phase approach and well enough to dominate lifecycle portfolio allocations as well as 100% equities.

To calculate the optimal consumption amount in each period would be a more complicated problem. But our interest is in the investment allocation. Given that consumption is chosen optimally, then the allocation of assets between stocks and bonds does not depend on the level of wealth (and hence doesn't depend on the amount of savings left over after consumption) and only depends on the relevant interest rate and the share of wealth that is liquid.

While the Samuelson framework was developed in a context where consumption was chosen optimally in each period, we can equally well apply this framework to a model where consumption is exogenously chosen during worklife. All of the portfolio risk is shifted to the retirement phase, so that consumption during retirement varies with the portfolio returns. While this is not optimal risk allocation, the assumption of exogenous consumption during worklife may fit the stylized facts for many workers with 401(k) plans, where workers tend to invest a constant fraction of their income each year.

4. Data and Methods

We simulate the returns from alternative investment strategies using long-term historical market data covering the years 1871–2004 collected by Robert Shiller (see Shiller 1989, 2005a, 2005b) and updated through 2006 using Global Financial Data. In order to include the returns to leveraged investment strategies, we add historical data on margin rates to the Shiller tables. For most of the analysis, we assume that the maximum leverage on

[15] The dollar amount invested in stock goes down for three reasons as the margin rate increases (above the risk free rate): (i) W decreases, (ii) λ decreases because of greater risk of leverage, (iii) λ decreases because of less diversification from censoring lower part of stock distribution.

stocks is 2:1, pursuant to the Federal Reserve Regulation T.[16]

For the margin rates, we use the broker "call money" rates.[17] This assumption may be controversial because many major brokers currently margin rates that are substantially higher than the current call money rate. For example, in May 2006, low-cost brokers such as Vanguard and Fidelity charged margin rates of more than 9.5% on small-balance margin loans, a rate that far exceeds their cost of funds.[18] The markups are independent of the degree of leverage and are instead tailored to the amount of the loan with substantial premiums for loans under $25,000 dollars. The corresponding margin rate at E*trade for loans over $1,000,000 was 6.74%, and Fidelity offered its active investors a rate of 5.5% on loans balances over $500,000. Several commentators (Fortune 2000; Willen and Kubler 2006) have noted that the high prices for small loan balances resemble credit-card rates more than asset-backed loans.

However, stock index derivatives have allowed investors to take on the equivalent of leveraged positions at implicit interest rates that are below the call money rate. Index futures, for example, are a more cost-effective means for most investors to take on a leveraged position. By placing 8% down as a non-interest bearing performance bond, an investor can purchase exposure to the non-dividend returns of all the major stock indexes. The standard equation relating the forward price to the spot price is:

$$F = S * e^{rT} - d,$$

where F is the forward price to be paid at time T, S is the spot price, d is any dividend of

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[16] The law independently limits the ability of individuals to invest savings on leveraged basis. Mutual funds offered inside and outside of defined contribution plans are limited in their ability to purchase stock on margin. Under the Investment Company Act, mutual funds registered as investment companies are prohibited to purchase "any security on margin, except such short-term credits as are necessary for the clearance of transactions." 15 U.S.C. § 801-12(a)(1).

[17] http://www.nber.org/databases/macrohistory/rectdata/13/docs/m13001.txt For 1971-2006, we use the Federal Reserves "prime loan rate." See also Global Financial Data. http://www.federalreserve.gov/releases/h15/data/Annual/H15_PRIME_NA.txt. The call money rate (also called the "broker call rate") is the interest rate that banks charge to brokers to finance margin loans to investors. See Fortune (2000) ("A widely used base lending rate is the broker call money rate, though some brokers use the bank prime rate.")

[18] Rates are as of May 1, 2006.

the underlying stocks, and r is the risk-free interest rate (Fortune, 2000). Using this equation (and accounting for the lost interest on the 8% performance bond), it is possible to back out an estimate of the implicit interest rate for constructing a leveraged position via stock index futures. Using forward and spot market data from 2000 to 2005, the implicit margin rate for the S&P 500 futures has averaged only 4.08%; see Table 1.[19] The implicit cost of borrowing is just 94 basis points above the average 1-month Libor rate for the same time period and is 174 basis points below the margin rates for the same time periods used in our simulations. This is an underestimate in that we have not increased the performance bond as would be required when stocks fall. Doubling the performance bond to 16% would increase the implied margin cost to 4.56%—still well below the "call money" rate at the time.

[Table 1 about here]

Table 1 also backs out an implicit interest rate for the UltraBull mutual fund. This fund employs a combination of options and futures to provide investors with twice the returns of the S&P 500 (i.e., a beta of 2). We calculate the implicit margin rate as the difference between twice the return on the S&P and the return on the UltraBull fund. For example, between 9/3/2002 and 8/20/2003, the S&P returned 13.93% while the UltraBull returned 22.89%; thus the implicit margin cost is 4.97%, the difference between double the S&P (27.86%) and the UltraBull return. Similarly, from 1/3/2001 to 12/25/2001, the S&P lost 15.06% while the UltraBull lost 34.99%, leading to an implied margin cost of 4.87%. Using returns data between 2000 through 2003, we find that the implicit interest is 5.09% or 1.6% above Libor, which is substantially cheaper than the rates offered by most retail brokers.

Even more direct evidence of the availability of low-cost interest can be seen from an

[19] The implicit interest rate may also be understated because owners of future indexes are subjected to less favorable tax treatment than owners of leveraged stock. Capital gains on future contracts are realized quarterly while realizations on stock investments may be deferred until a stock sale. IRS rules mitigate this difference by allowing holders of future contracts to attribute 60% of income as long-term gains and 40% as short-term gains.

analysis of deep-in-the-money Leap call options. For example, on July 6, 2005, when the S&P 500 Index was trading at $1194.94, a one-year LEAP call option on the S&P index with a strike price of $600 cost $596.40. This contract provides almost 2:1 leverage. It allows the investor, in effect, to borrow $598.54 (as this is the savings compared to buying the actual S&P index). To earn the same return as the S&P, one would have to pay the exercise price ($600) plus foregone dividends ($22.44). The total cost of paying $622.44 almost a year after borrowing $598.54 produces an implied interest of 3.78% which is 25 basis points over the contemporaneous one-year yield on a treasury note. Table 2 derives the implied interest of thousands of LEAP call options for ten years of option data.

[Table 2 about here]

We find, for example, that the implied interest for deep-in-the-money call options that produce effective leverage between 200 to 300% averaged less than one percent above the contemporaneous 1-year Treasury note. Moreover, the average implicit interest rate on these calls was 160 basis points below the average contemporaneous call money rate.

However, Table 2 also shows that the implicit interest increases with the degree of leverage. The marginal interest of moving from 3:1 to 4:1 leverage is substantially higher than the increase in average implicit interest. We find that the marginal cost of increasing leverage rises sufficiently that it is unlikely that it would be cost effective to invest at leverage of more than 3:1 via option contracts.

The more important lesson of Tables 1 and 2 is that the derivative markets have made it increasingly inexpensive to invest 200% or even 300% of current saving accumulations in the stock market. Whether or not investors had ready access to the broker "call money" rate in the past, our assumption of low-cost money going forward is particularly reasonable given the advent of options to implicitly borrow through derivative markets.

Table 3 shows summary statistics for the nominal financial returns. Stocks over this period had an average nominal return of 9 percent. The maximum positive return was 51.4% in 1933 shortly after the maximum negative return of −26.2% in 1931.[20]

[Table 3 about here]

Using Shiller's monthly data on stock and bond returns from 1871 to 2004, updated to 2006, we construct 93 separate draws of a worker's 44-year experience in the markets. Each of the draws represents a cohort of workers who are assumed to begin working at age 21 and retire at 65. For example, the first cohort relates to workers born in 1850 who start to work in 1871 and retire in 1915.

To perform the simulations, we take a representative worker and imagine that individual has an equal chance of experiencing any of the 93 different return histories. (Later, we also allow the worker to randomly experience returns from any 44 years out of the 135 in our total sample.) Following PRVW (2005b) and Shiller (2005a), we assume that workers save a fixed percentage of their income. In our simulations, we use Shiller's 4% number. Thus the saving accumulations depend only on the history of 4% contributions and prior-year returns.

Although the percent is constant, the actual contributions depend on the wage profile. We assume a hump-shaped vector of annual earnings taken from the Social Security Administration's "scaled medium earner." Wages rise to a maximum of $58,782 at age 51 (generating a saving contribution in that year of $2,351) and then fall off in succeeding years.[21] For a new worker at age 21, the future saving stream has a present value of $44,837 (when discounted at a real risk-free rate of 2.7%). The humped flow of saving contributions along with the present value of future contributions are shown in Figure 1. Given this flow of saving contributions, the simulation assesses how different investment

[20] Our simulations are based on real returns and real interest rates. However, when we later consider the potential impact of margin calls, we employ nominal returns as margin calls depend on the nominal change in equity prices.

[21] See Shiller 2005a, Clingman and Nichols (2004), http://www.ssa.gov/OACT/NOTES/ran3/an2004-3.html, Table 6, (scaled factors); http://www.ssa.gov/OACT/TR/TR04/lr6F7-2.html (average wage).

strategies fare in producing retirement wealth. In performing these calculations, we assume an annual administration/transaction fees equal to 30 basis points of the net portfolio value.

[Figure 1 about here]

4. Using Simulations to Complete the Model

To complete the model, we need to derive the percentage targets for specific levels of constant relative risk aversion (γ). To do this, we first find the dual-targets—a leveraged (λ_a) and unleveraged (λ_b)—that maximize single-period expected utility using the sample 135 returns as the actual distribution of returns.

Because the utility function is multiplicative in returns, maximizing single-period expected utility is equivalent to choosing the equity allocation to maximize $\frac{E\left[R^{1-\gamma}\right]}{1-\gamma}$, where R is the resulting blended return. In the case of the leveraged target, we use the margin rate as the opportunity cost of capital; in the case of the unleveraged target, we use the government bond rate. (The general formula for R is provided in the appendix, Equation 1.) Our point to emphasize is that we chose the equity allocations to maximize single-period expected utilities according to the historical distribution of returns; we did not choose the allocations so as to maximize the ex-post lifetime utilities of the 93 cohorts.

The results from this maximization are shown in Table 4. For CRRA = 2, the optimal leveraged and unleveraged percentage targets are 88.0% and 91.6% respectively. These percentages form the core example that we evaluate in our simulation of the dual-target strategy.

[Table 4 about here]

While we expect the unleveraged percentage target to be higher than the leveraged percentage, these two percentage targets are very close. This is because (as seen in Table 3) the average margin rate in our data is only slightly higher than the average corporate bond rate, 5 percent versus 4.8 percent. This leads us to evaluate a single-target (three-phase) strategy, which invests a constant 88.0% of wealth, subject only to maximum leverage constraints.

We focus our attention on two different temporally diversified strategies and compare them with the two traditional investment strategies. Specifically, our simulations compare:

1. Dual-Target (Four-Phase) Strategy. This strategy sets the initial equity percentage target at a lower percentage (88.0%) during the first and second phases of leveraged investment and at a higher percentage (91.6%) for the third and fourth phase of unleveraged investments.

2. Single-Target (Three-Phase) Strategy. This strategy sets the equity percentage target at a constant percentage (88.0%) of discounted savings. Initially, the worker invests her entire liquid savings on a fully leveraged basis of 2:1 and remains fully leveraged until doing so would create stock investments exceeding the target percentage. From then on the worker invests on a partially leveraged or unleveraged basis. If the unleveraged portfolio value exceeds the target percentage, then stocks are sold and the excess amount is invested in government bonds. The percent of the portfolio invested in stock is contingent on the prior year realized returns as this impacts the current portfolio value.

3. 100% Stock. Under this benchmark strategy, the worker invests a constant 100% of her liquid savings in stock.

4. 90%/50% Life-Cycle. Under this benchmark strategy (based on the Fidelity Freedom fund allocation) the worker invests 90% of portfolio value in stock at age 21 and the percentage invested in stock falls slowly to 82% by age 44 and then falls more rapidly to 50% by age 65.

We limit our comparison set to these two traditional investment strategies in order to conserve space. PRVW(2005b) and Shiller (2005a) have simulated the risk and return of more than a dozen traditional investment strategies—included 100% TIPS, 100% bonds, (110 − Age)% in stocks and a variety of alternative life-cycle strategies.

5. Results of the Cohort Simulation

a. Deviations from Optimal Diversification

From a diversification perspective, there are two problems with traditional investment strategies. The front-end problem is that the strategies don't expose the worker to sufficient stock market risk—thus throwing away the potential for additional years of diversification. The back-end problem is that strategies tend to expose the worker to either too much risk (under the 100% rule) or too little market risk (under the 90/50 rule).

To provide some heuristic evidence about the size of the front-end and back-end failures to diversify, we estimate the average amount invested in stock for the four benchmark strategies. A temporally diversified strategy would maintain a constant percent of retirement wealth in equities. Since retirement wealth grows at the blended return, if all wealth were available up front, we would also expect to see equity investments growing at the blended return rate, here 6.28% assuming that CRRA=2.

Of course, the optimal temporal diversification will also depend on liquidity constraints, the cost of margin borrowing and on the realized returns in prior years, but it is valuable, heuristically, to see how close traditional strategies come to the Samuelson ideal.

Figure 2 shows the average present value invested in stock in each year of the investor's working years for the 93 worker cohorts.

[Figure 2 about here]

Both the 90/50 and the 100% strategies fail to invest substantial amounts in stock in the first quarter of the investor's working life—effectively discarding these years as a means to diversify stock market risk. The leveraged diversification strategies respond directly to this problem by investing more in stock and thus putting the initial investor on a much steeper slope of investments. The back-end problems are even more pronounced. The 100% investment has the expected result of exponentially increasing the amounts invested in stock so that the returns in the few final years alone will disproportionately impact the investors' retirement wealth.

The 90/50 lifecycle exhibits the alternative back-end problem of not investing enough in stock in the last working years. Overall, the 90/50 strategy achieves a relatively flat real exposure to the market from age 45 onwards. But this is done at a cost of too little overall exposure. The lifecycle fund only has a 65% average exposure to the market. Our single-target (88%) strategy achieves a little over 100% exposure, but mitigates risk by achieving better diversification across time.[22]

We can also assess the extent of diversification by measuring the concentration of strategy's exposure to stock market risk. The reciprocal of the Herfindahl-Hirshman Index (HHI) is a heuristic measure of the effective number of diversification years. Just as the inverse of the HHI in antitrust indicates the effective number of equally-sized investors in an industry (Ayres 1989), the inverse of the HHI here indicates the amount of diversification that could be achieved by investing equal dollar amount in separate years. HHI estimates indicate that the average worker using the 100%-investment rule effectively takes advantage of only about 27.8 of her 44 investments years (63.3%). In

[22] The initial 200% exposure which ramps down to 88% still averages 100% because of the greater size of the portfolio in later years.

contrast, the single target strategy takes advantage of 33.1 years (75.2%). As seen in Figure 2, under the 90/50 rule, the worker's exposure to stock market risk is more evenly distributed across years—and the inverse HHI in turn increases to 33.9 years (77.7%). But this increase in effective diversification is achieved by generally limiting exposure to the stock market. Investing nothing in stocks each year likewise would be fully diversified.

b. Comparing the Five Investment Strategies

Table 5 reports our core results. In it, one can see the distribution of retirement wealth for the 93 worker cohorts under the five investment strategies. The first two columns replicate the basic findings of Shiller (2005a) and PRVW(2005) in showing that a simple strategy of investing 100% of accumulated savings in stock dominates the life-cycle strategy of investing 90% in stock when young ramping down to 50% at 65. Average retirement wealth among the 93 cohorts is more than 53% larger with the 100% strategy ($376,554) than with the 90/50% strategy ($244,989) and the certainty-equivalent dollar amounts are uniformly higher for all reasonable relative risk aversion measures.

[Table 5 about here]

The surprise is how well the leveraged strategies fare relative to the 100% strategy. The dual-targets produce a median retirement wealth that is 27.6% higher than the 100%-stock strategy and an increase in the mean return of 21.1%.[23]

The higher returns of leverage do not, however, translate into higher retirement risk. The minimum retirement wealth under the dual-target strategy was 13.6% higher than the minimum return of under the 100% stock strategy—and the 10th percentile was 23.3%

[23] Workers using this strategy would not immediately invest the targeted amount in stock because they would be constrained by the 2:1 legal leveraging limit. Table 5 indicates that the average amount of leverage over the course of the working life (weighted by the present value invested in stock) was only 104.6%.The 2:1 leverage is on small dollar amounts, and when averaged with the unleveraged investments before retirement, the present value-weighted average leverage is very low.

higher. Table 5 shows that the mean, median, minimum, maximum, 10^{th}, 25^{th}, 75^{th}, and 90^{th} percentiles for both the dual and the single-target strategies are all higher than those of the 100% stock strategy. Moreover, the lower panel of Table 5 shows that the certainty-equivalent dollar values for retirement wealth are 13% to 22% larger for both the dual-target strategy and the single-target strategy compared to the 100% stock strategy.[24]

As seen in Table 5, a single-target produces substantially similar results as the dual-target. We expect that dual-target strategies will do better when margin costs are important, but empirically most of the benefits of temporal diversification can be achieved with a single target. The single target has the added benefit of simplicity and so, the remainder of the paper will focus on the single-target strategy.

Table 6 shows the median length of the different phases. For the single-target (88%) strategy, the investor in the median cohort is maximally leveraged until age 32 and continues to have some degree of leverage until age 50.

[Table 6 about here]

The advantage of the single-target strategy is most clearly seen in Figure 3. The single-target strategy stochastically dominates the return of both conventional investment strategies. First-order stochastic dominance can be seen by the fact that cumulative distribution functions for the 93 investing cohorts are everywhere to the right.

[Figure 3 about here]

One concern is that the stochastic dominance of the single target strategy comes from its higher overall exposure to the stock market and not from any diversification advantage. From Table 5, we know that the average percent invested in the stock market (weighted

[24] Note that the investment strategy was based on CRRA = 2. Thus for the other values of CRRA, the expected utility would have been even higher had the strategy been reoptimized.

by the present value invested in the market each year) is higher for the 88% strategy than for either of the traditional strategies. But Table 7 shows that a less aggressive but still leveraged strategy that has the same average exposure to stock will substantially reduce risk. A 76.7% leveraged strategy (which starts at 2:1 leverage and eventually ramps down to 76.7% invested in stock) on average invests the same percent in the stock market as the traditional 100% strategy. Table 7 shows that the 76.7% leveraged strategy is substantially less risky. The minimum and 10th percentile cohort returns increase by 9 and 19 percent relative to the traditional 100% strategy, while the maximum and 90th percentile returns fall by 13 and 10 percent respectively. However, the means were not quite the same due to the timing of historical returns. Thus we further adjusted the leveraged strategy to a target of 74.6% to achieve equal mean returns. Here the minimum, 10th, and 25th percentile were all higher, while the 75th, 90th, and maximum returns were all lower. By spreading investments more evenly over time, we see that a leveraged strategy can generate a mean-preserving reduction in spread.

[Table 7 about here]

PRVW (2005b) showed that life-cycle strategies were largely equivalent to investing constant fraction of current savings in stock market. But our second order dominance table shows that a life-cycle strategy that does a better job of diversifying over time can stochastically dominate investing constant fraction of current savings. While the standard deviation of returns for leveraged strategies in Table 5 increased relative to those for the traditional strategies, Table 7 shows that leveraged strategies exist which can preserve the mean returns and reduce the standard deviation by more than 20%. This table concretely shows the potential benefits to temporal diversification.

We also conducted a paired cohort-by-cohort comparison of temporally diversified and traditional investment strategies. Table 8 shows that the single- and dual-target accumulations were higher than the 90/50% strategy in all 93 cohorts and better than the 100% stock strategy in 90.3% (84 out of 93) of the cohorts. A sign test finds these proportions to be statistically different than 50% ($p \leq 0001$).

[Table 8 about here]

The 9 cohorts where the 100% strategy beats the single-target strategy all occurred for the most recent retiring cohorts (1999–2007). We were initially concerned that we were recommending that people consider single-target strategies just when strategy was starting to fair more poorly. A closer investigation of the recent results (shown in Figure 4) suggests that single-target strategy fell behind the 100% strategy because the single target investors did not invest as aggressively in the stock market in the 1990s during the historic run up (for example, a nominal 32% increase in 1991). The 100% dominated the 88% single-target strategy because the latter was more conservative in the investors' later years.[25] The *relative* shortfall of the single-target strategy was not, however, an absolute shortfall. All nine cohorts in which the 100% strategy exceeded the single-target strategy are cohorts where the single-target strategy produced above-average accumulations—but just not quite as high as the 100% strategy because they slightly ramped down the stock allocation in the last phase before retirement.

[Figure 4 about here]

c. Margin Calls and Wipeouts

The analysis underlying Table 5 did not allow for interim margin calls that would occur if there was a substantial decline in the market. Our estimation assumed that all leveraged positions were closed at the end of each month and, if the strategy ordained, releveraged up in the next month. The major stock exchanges (per NYSE Rule 431 and NASD Rule 2520) require a maintenance margin on long positions of 25%. Some brokers require an even higher maintenance margin of 30% or 35% (Fortune 2000).

If there were no maintenance margin requirement, the stock market would have to drop

[25] In 2000-2003, the market declined annually 5.5, 13.1 and 20.0 percent. For investors retiring just after these years it was good not to be so invested and accordingly the shortfall in accumulations narrows in Figure 4.

50% before the net value in a fully (200%) leveraged portfolio was extinguished. But with a maintenance margin requirement of 25%, margin calls would force investors to start selling their positions if the market lost a third of its value.[26] Margin calls do not greatly affect our analysis. They merely force the investor to delever the portfolio by selling some of their stock and retiring some of their debt. Being forced to delever in June can reduce your returns if the market rebounds by the end of the year. But being forced to delever can also increase your returns if the market further deteriorates. Table 9 shows that (as random walk theorists would predict) months with margin calls are followed by a mixture of rebounds and deteriorations. Hence not adjusting our estimates for margin calls caused by stock swings within particular months, while a departure from reality, should not bias or even greatly impact the results.

[Table 9 about here]

Even without margin maintenance requirements, the prevalence of substantial market declines has a potentially devastating impact on strategies that incorporate leveraged stock purchases. A natural reality check is look at the results for worker cohorts who lived through the depression years. The real stock returns on the S&P 500 in 1929, 1930 and 1931 were −8.8%, −16.0%, and −36.5%.[27] How can it be that investors following leveraged strategies did as well as reported in Table 5? The basic answer is that workers who retired just after the crash were not severely hurt because the targeted strategy had already eliminated their leverage. For example, workers retiring in 1932 following the single-target strategy would have had just 89% of their portfolio invested in the market when the market lost more than a third of its value. Because of the success of their investments in previous years, they would still have a retirement wealth of $249,435, still slightly above the average result reported in Table 5 for the conventional 90/50 investment strategy.

[26] Imagine that the investor buys $200 of stock using $100 of capital. Were the market to drop by 33.3%, then the portfolio would be worth $133 and the equity behind it would be $33.3 or 25%.
[27] The stock market "crash" in October 1929 had been preceded by sizable increases so that the year-end nominal loss was only 8.8%.

Individuals who began working just before the depression adopting the single-target strategy would have done even better. Those who entered the labor force in 1931 immediately have experienced a 86.5% loss in their first investment year. But this is a large percentage of a small amount, and the target strategy responds by keeping these workers fully leveraged until they hit the target. By the time of their retirement in 1975, these workers following the single-target strategy would have accumulated $410,440 (in 2006 dollars), which is well above the median return for the 100% stock strategy.

The single-target strategy produced the lowest accumulations for workers retiring in 1921 ($149,106). For these workers, enduring the double digit market declines in 1893, 1903, 1907, 1917, and 1920 was more limiting than the more severe, but compact, declines of the depression.

As an empirical matter, we see in our monthly data that the stock market has never declined sufficiently to wipeout the preexisting investments of any cohort adopting a temporally diversified (single- or dual-target) strategy. Table 10 details the prevalence of negative monthly returns for the 93 cohorts over their 528 months of investment. The leveraged single-target strategy would have produced negative returns of 53% in October of 1929 for young investors who are fully leveraged (2:1). Leveraged strategies expose workers to a much larger probability of incurring a substantial negative monthly return sometime during their working life. Roughly one-third of the cohorts (28 out of 93) would have lost over 40% in one month. Table 5 shows, however, that exposure to a substantial risk of a substantial monthly loss does not mean exposure to a substantial risk of loss to accumulated retirement savings.

[Table 10 about here]

d. Alternative Margin Caps

While Regulation T prohibits investing more than 200% of portfolio value in stock, absent this regulation, lenders might agree to higher degrees of leverage. Home mortgages are usually much more leveraged and secured by non-callable and less liquid security. In fact, current stock index future contracts require only about an 8% "performance bond" and thereby allow qualified individuals to invest on the order of 1,250% of their equity value. Table 11 analyzes the impact of higher margin caps on the single target (88%) leveraged investment strategy. The two left-hand columns report as benchmarks the accumulations from the more traditional 100% stock strategy as well as the single-target strategy for a 200% margin cap.

[Table 11 about here]

If leverage caps were increased to 300% or 500%, the single-target strategy would have been even more successful. Mean, median and minimum accumulations are higher for the 500% cap than for the 200% cap. For example, the mean retirement accumulation is $497,610 under the 500% cap instead of $448,037 under the 200% cap. Moreover, the certainty equivalents for the 500% are substantially higher than for lower caps. The CRRA = 2 certainty equivalent is $417,426 for an investment strategy with a 500% cap but only $380,881 for investments with a 200% cap—an improvement of 9.6%. This is true even though the 500% strategy in 11 separate cohort months (out of 49,104 cohort months) produces annual returns of −100% which completely extinguish the cohort's preexisting retirement accumulation.[28] Even with 5:1 leverage, the minimum performance is 40% higher than under the traditional all-stock strategy. The take-home lesson of Tables 10 and 11 is that the existence of substantial short term risk—even the risk of losing everything—does not undermine the expected gain from a disciplined, leveraged investment strategy.

6. Robustness

[28] Nine of the wipeouts were associated with cohorts where youthful investors were fully leveraged in March 1932 (when the S&P 500 fell 23%) and two wipeouts were associated with cohorts where youthful investors were fully leveraged in October 1929 (when the S&P 500 fell 26%). Even with a 500% cap, October 1929 did not effect more cohorts because the high returns in roaring 20's had already reduced investors leverage heading into the crash.

This section considers alternative assumptions to test the robustness of the advantages to leveraged investing. We consider higher margin costs, as well as lower stock returns. We consider simulations based on foreign stock returns. We also redo our analysis using Monte Carlo simulations with replacement. The consistent message is that our results are robust to a variety of assumptions. This is foreshadowed by our summary statistics in Table 3. From 1871 to 2006, the average premium on stocks over the margin rate was 4 percent—9% vs. 5%. As long as the expected return on stock exceeds the net cost of maintaining a margin position, it will be optimal to employ leverage early in life. As the premium narrows, the scale and value of leverage declines.

a. Higher Margin Rates

Higher margin rates narrow the equity premium when buying stock on margin and thus reduce the value of leverage. We have assumed the margin rate averaged just 20 basis point above the return on corporate bonds (5.0% vs. 4.8%, as shown in Table 3).

Table 12 reports the impact of simply increasing the historic margin rates. The two left-hand columns of Table 12 report the benchmark accumulations accruing to the 100% and single-target strategies. The next four columns report the statistics for the single-target strategy where the margin loan rate is raised by 1–2.5%.

[Table 12 about here]

Table 12 shows that the median and mean returns increase substantially even with 250 basis points added to the historic margin rates used in Table 5. The optimal percentage target is a function of both the individual's risk aversion and the expected risk of stock investment—including the risk of leveraged investments in stock. As the cost of leverage increases, the optimal percentage target for any given CRRA would decrease. But Table 12 shows that, even without adjusting down the target percentage to account for the higher cost of leverage, it is still possible to produce superior accumulations. As the margin rates increase by 200 basis points, however, the (non-optimized) 88% strategy becomes riskier and produces fewer benefits relative to the unleveraged 100% stock strategy for very risk-averse investors (CRRA above 4).

As theory would predict, the diversification advantage of leveraged investment strategies is contingent on the cost of borrowing. Yet Table 12 shows that even an invariant leveraged strategy dominates the 100% stock strategy for margin rates up to 250 basis points above the corporate bond rate. The effective cost of leveraging through stock index contracts is far below this cutoff.

b. Lower Stock Returns

Leveraged strategies will also be less attractive if the expected return on stocks is lower. Shiller (2005a) has suggested several reasons why the success of U.S. stocks in the 20th century will not be replicated in the 21st. He shows that the returns on stock in other countries has been 2.2% lower than the stock returns in the U.S. Brown, Goetzmann and Jorion (1999) find an even larger shortfall. Moreover, a 2005 *Wall St. Journal* survey of prominent economists at Wall Street brokerages reports an expected real stock return of just 4.6%, which is 2.2% lower than the return found in the historic (1871–2006) data.

Unlike higher margin costs which just impact the expected return of leveraged strategies, the possibility of lower stock returns also impacts the expected accumulation of unleveraged investment strategies. Accordingly, Table 13 reports the results of reducing the nominal annual stock return by various percentage points for both the 100% stock strategy and single-target strategy.

[Table 13 about here]

Table 13 shows that the single-target (88%) strategy produces higher means and medians even with lower stock returns. With 1.5 percentage points subtracted from stock returns, the median retirement accumulation is 20.1% higher ($255,266 vs. $212,498) and even with a 2.5 percentage point reduction, the median accumulation is 19.4% higher ($186,921 vs. $156,523). The single-target strategy produces a slightly lower minimum return (0.1%) than the 100% stock strategy when 2.5% is subtracted from the annual stock returns. However, for relative risk aversions of 2, 4, and 8, the certainty equivalent for the single-target strategy is still 3.9 to 7.4% higher than that for the 100% strategy—even when 250 basis points is subtracted from the stock returns. As with increased margin rates, the optimal percentage target would decline with lower expected stock premia. But Table 13 shows that, even without reoptimizing, the advantages of the leveraged 88% investment strategy are robust to a substantial fall in the equity premium.

c. Foreign Returns

We also investigated how the single-target (88%) strategy would have fared in other parts of the world relative to the traditional 100% strategy. Table 14 reports the results of an analogous cohort exercise using monthly returns on the FTSE (1937–2007) and Nikkei (1956–2007). For the FTSE All-Shares Index, we find in 26 cohorts that the single-target strategy produced mean and median returns that were 22 and 25 percent higher than the traditional 100% strategy and a minimum return that was 40.7% higher. For the Nikkei Index the advantage of the leveraged strategy was more modest—the mean return and certainty equivalent increase is on the order of 14%. But even without reoptimizing the single-target percentage for the expected Nikkei return distribution, we were once again able to produce higher certainty equivalent.

[Table 14 about here]

d. Monte Carlo Simulations

An advantage of the cohort simulations is that they tell what actual investors might have achieved in the past if they had pursued our proposed investment strategies. But the 93 cohorts analyzed in Table 5 are clearly not independent of each other. The returns of any two adjacent cohorts massively overlap—so that our effective number of independent observations is closer to 3 [≈(2006−1871)/44]. An alternative approach to estimation pursued by PVRK (2005) is to use the historic returns as the basis for a Monte Carlo simulation in which workers randomly draw returns with replacement from an urn of the yearly returns. We estimate the distribution of returns from 10,000 trials, each time picking 44 years at random from Shiller's annual data with replacement.[29] This approach produces returns that are independent and identically distributed—even though it is not clear that the stock returns are in fact independently distributed across time (Poterba and Summers 1988). One thing is clear: leverage strategies no longer produce first-order stochastic dominance. The reason is that with a large enough sample, some workers will draw the 1931 returns 44 years in a row. If nature draws depression many times in an investor's life, unleveraged strategies will do better.

The results of the Monte Carlo simulations are reported in Table 15. The leveraged single- and dual-target strategies continue to produce higher mean and median returns than either of the traditional investment strategies.

As predicted, the absolute minimum return was substantially lower for Monte Carlo with replacement than with the cohort analysis. For the 10,000 simulations, the minimum return came from a draw that in quick succession had three depression years: two 1930's and one 1929). Even the presence of this rare event did not cause the certainty equivalents (or the 10th percentile returns) for the single-target strategy to be lower than the traditional strategies.

But Table 15 also shows that the CRRA-invariant leveraged strategies do not produce uniformly higher certainty equivalents. For CRRAs equal to 4 and above, the traditional, unleveraged strategies produce higher certainty equivalents. The 88% strategy, however, was optimized for an investor CRRA equal to 2. Table 5 showed that, for the historical data, invariant percentage targets still produced higher certainty equivalents than the traditional investment strategies, even for very high levels of risk aversion. In contrast, Table 15 shows that under Monte Carlo simulation, the certainty equivalents for invariant

[29] Monte Carlo with replacement subjects investors to riskier i.i.d. returns. Like the cohort analysis, the draws from these Monte Carlo simulations without replacement are not i.i.d. Once an investor has drawn 1929, she never has to worry about hitting it again.

targets can become substantially lower than the traditional strategies when risk aversion rises. Investors with higher levels of risk aversion should pursue leveraged strategies with lower targets.

[Table 15 about here]

To investigate the impact of higher degrees of risk aversion, we reanalyzed the relative returns using the single percent targets (reported earlier in Table 4) that are reoptimized for particular degrees of risk aversion. Table 16 reports the certainty equivalents for these optimized percent targets. We see that for CRRA = 2, the optimal single percent target remains at 88.0%. But, for higher levels of risk-aversion, the optimal percent target decreases. Table 16 shows that using CRRA-specific targets once again produces certainty equivalents that substantially exceed those of both the traditional 90/50 and 100% strategies. In the historic data, the benefits of temporal diversification were so great that the CRRA-invariant targets were sufficient to generate gains. With Monte Carlo simulations, temporal diversification still produces benefits but CRRA-specific targets must be used.

[Table 16 about here]

e. Diversifying Across Time versus Stocks

From a dynamic perspective, investing on margin reduces risk because it allows the investor to better diversify risk across time. Diversifying across time and across assets are the only two dimensions on which diversification is possible. Indeed, temporal diversification is more important because returns across different years tend to be less correlated than returns across different stocks within any given year. If only one type of diversification were possible, diversification across time lowers risk more than across stocks.

Table 17 shows the comparative strength of asset and temporal diversification by comparing the distribution of returns from full asset diversification for a single random year out of 21 years to the return distribution from investing 1/21st of your portfolio each year in a single stock. The mean returns are nearly identical, but the temporal diversification produces substantially less variation in returns.

[Table 17 about here]

7. Conclusion

This paper shows that it is possible for people to retire with substantially larger and safer retirement accumulations, and they can do this without having to save more. All they have to do is invest using leverage while young. Our result puts into practice Samuelson's original insight that people with constant relative risk aversion should invest a constant percentage of their lifetime wealth each period in stock. For young workers, wealth exceeds liquid assets. Thus to implement the Samuelson rule requires leveraged purchases when workers are young.

Our recommended investment strategy is simple to follow. An investor who targets a single percentage or a single present dollar value follows three phases of investment. The worker begins by investing 200% of current savings in stock until a target level of investment is achieved. In the second phase, the worker maintains the target level of equity investment while deleveraging the portfolio and then maintains that target level as an unleveraged position in the third and final phase.

The expected gains from such leveraged savings are striking. With increased longevity, people need to save more for their retirement. The expected gains in retirement accumulations relative to the traditional 90/50 lifecycle strategy would allow someone to finance an extra 21 years of retirement (past age 100) or to retire at age 56 and still finance retirement through age 85. Or, to the extent that current savings are inadequate to maintain pre-retirement standards of living, this can boost retirement consumption by 65%.

Our results depend on historical factors that may not repeat. Most importantly, our results depend on the equity premium. For typical levels of risk aversion, the advantages of a leveraged strategy are reduced but continue to hold even if the equity premium were to fall by nearly 250 basis points. The estimation does not take into account the impact of non-portfolio wealth, such as housing and human capital. Workers with non-portfolio wealth that is correlated with the stock market already have some elevated exposure to stock market risk. Thus the target level of equity holdings should include the human capital exposure to the market. The relevance of this issue will vary across professions and is a subject for future investigations.

Finally, our results have significant implications for legal reform. The natural places to engage in leveraged purchases are IRA and 401(k) accounts. Yet, with the exception of the index options, leveraged and derivative investments inside these accounts are prohibited. An employer who offered workers the option of following our leveraged single-target strategy might risk losing their statutory safe harbor. Approximately two-thirds of 401(k) plans allow employees to borrow against their plan balances to fund present consumption; in stark contrast, employees are not allowed to borrow to fund leveraged investments for their future. Young workers with non-tax deferred retirement savings can lever their net retirement portfolio with the use of stock index futures, but even here the law intrudes limiting future accounts to investors who are "sophisticated" (which is often means little more than sufficiently rich).

The legal constraints are not the primary reason that people fail to buy enough stock when they are young. Despite compelling theory and empiricism, many people have a strong psychological aversion to mortgaging their retirement savings. While families are encouraged to buy a house on margin, they are discouraged and often prohibited from buying equities on margin. We are taught to think of leverage investments as having the goal of short-term speculation instead of long-term diversification. As a result, most people have too little diversification across time and too little exposure to the market when young. Based on historical data, the cost of these mistakes is substantial.

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Appendix

Here we calculate the portfolio rate of return. The margin coverage requirement is denoted by *m*.[30] For each dollar in stocks, the investor must put up *m* dollars in cash. Thus the maximum fraction of wealth that can be invested in stocks is λ/m, where λ is the unleveraged share of wealth invested in stocks. Without loss of generality, we assume that the person maximizes her ability to borrow stocks on margin. To the extent that she doesn't want to borrow money to buy stocks on margin, the person "invests" that money back in a bond that pays the margin rate of interest, r_m. In essence, when the person invests in bonds that pay the margin rate of interest, it is as if she is borrowing less. If the fraction of wealth invested in stocks falls below 1, then the residual is invested in bonds paying the risk-free rate $r_f \leq r_m$.

Let *z* be the return on equities. The overall return to the portfolio, *R*, is:

$$R = \frac{\lambda}{m} * z - \max[\frac{\lambda}{m} - 1, 0] * (1 + r_m) + \max[1 - \frac{\lambda}{m}, 0] * (1 + r_f). \quad (1)$$

There is a discontinuity in the relevant interest rate at $\lambda = m$. Until that point, the investor is buying stock on margin and thus faces an opportunity cost of $1+r_m$. Once $\lambda = m$, the investments are made on an unleveraged basis—so the opportunity cost to buy additional stock is $1+r_f$.

[30] The coverage rate is determined by regulation and brokerage firms. It is not a choice variable. If the coverage requirement were 40% and the investor were to put 60% of her cash into stocks, that would allow her to buy stocks worth 60/40 = 150% of her initial cash. In practice, the initial margin coverage is larger than the maintenance coverage level and we control for this complication in our simulation.

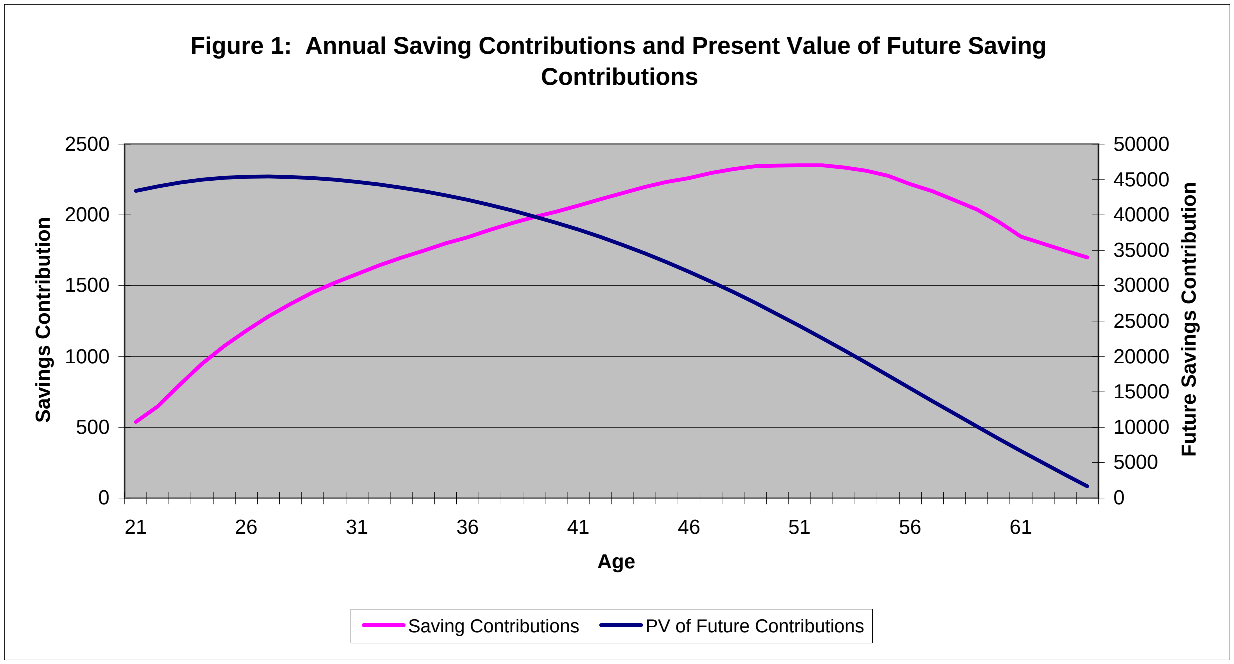

Figure 1: Annual Saving Contributions and Present Value of Future Saving Contributions
Savings Contribution
2500
2000
1500
1000
500
0
50000
45000
40000
35000
30000
25000
20000
15000
10000
5000
0
Future Savings Contribution
21
26
31
36
41
46
51
56
61
Age
Saving Contributions
PV of Future Contributions

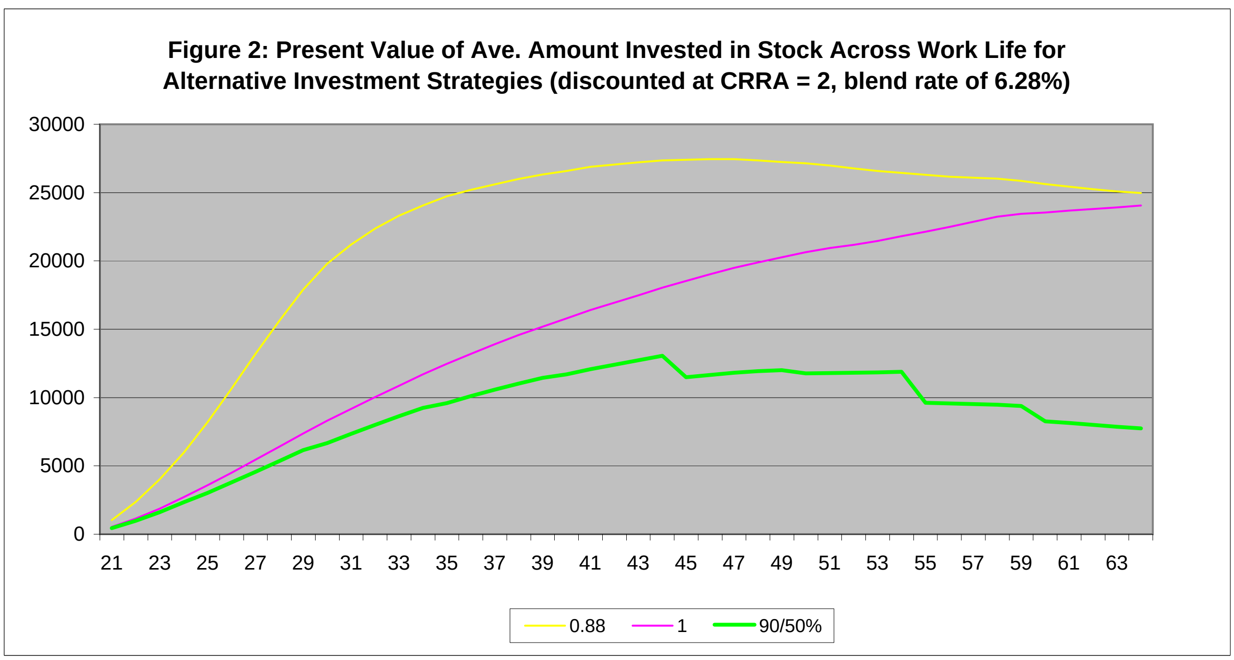

Figure 2: Present Value of Ave. Amount Invested in Stock Across Work Life for Alternative Investment Strategies (discounted at CRRA = 2, blend rate of 6.28%)
30000
25000
20000
15000
10000
5000
0
21
23
25
27
29
31
33
35
37
39
41
43
45
47
49
51
53
55
57
59
61
63
0.88
1
90/50%

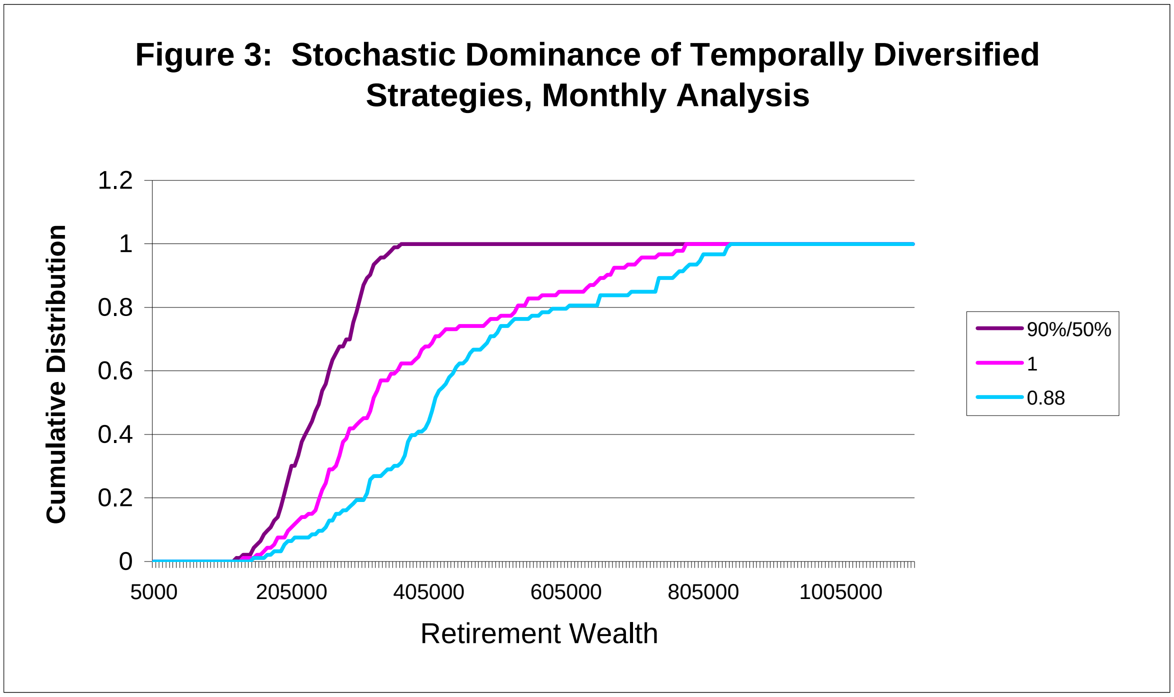
Figure 3: Stochastic Dominance of Temporally Diversified Strategies, Monthly Analysis
Cumulative Distribution
1.2
1
0.8
0.6
0.4
0.2
0
5000
205000
405000
605000
805000
1005000
Retirement Wealth
90%/50%
1
0.88

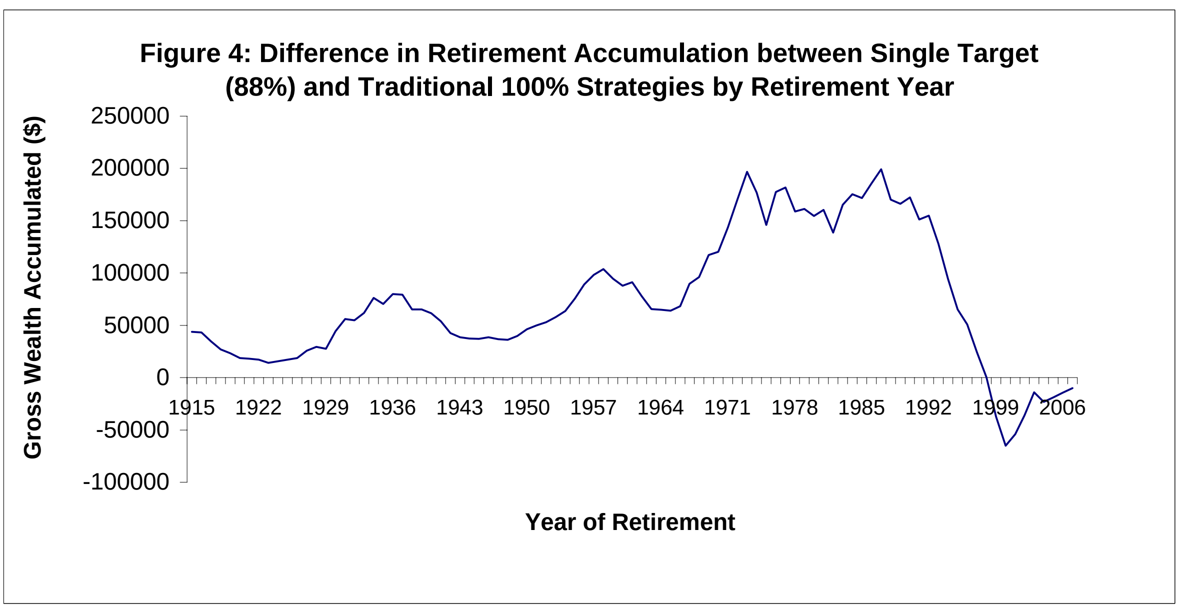
Figure 4: Difference in Retirement Accumulation between Single Target (88%) and Traditional 100% Strategies by Retirement Year
Gross Wealth Accumulated ($)
250000
200000
150000
100000
50000
0
-50000
-100000
1915
1922
1929
1936
1943
1950
1957
1964
1971
1978
1985
1992
1999
2006
Year of Retirement

Table 1: The Implicit Costs of Borrowing Via Stock Index Futures and UltraBull Mutual Fund

Average Implicit Borrowing Rate

S&P 500 Future*	Margin Rate	1 Month. Libor	Prime Rate
4.08%	5.82%	3.14%	6.02%
UltraBull Profund**	Margin Rate	1 Year Libor	Prime Rate
5.09%	5.29%	3.49%	5.39%

*Average Implicit Annualized Interest Rate for daily future and spot data from 1/1/00 - 5/31/06. All data (including S&P spot and future prices and dividend yield from Global Financial Data.
**Average Implicit Annualized Interest Rate for 10 overlapping year-long periods between 1/01 and 10/03. Data from www.profunds.com

Table 2 : Implied Interest Rates - 1 Yr. S&P Calls 1996-2006

Range of "Leverage" Ratios	Contracts Observed	Average Implied Interest Rate	Mean Spread Over Margin Rate	Mean Spread over 1-Year Treasury Note	Marginal Interest Rate at Mean Spread
1 - 2	347	4.02%	-1.96%	0.53%	-
2 - 3	1,857	5.07%	-1.68%	0.92%	4.92%
3 - 4	1,998	5.70%	-0.93%	1.76%	6.60%
4 - 5	1,794	6.67%	-0.01%	2.73%	8.66%
5 - 6	1,485	7.61%	0.93%	3.67%	10.50%
6 - 7	1,281	8.54%	1.78%	4.53%	11.96%
7 - 8	1,022	9.18%	2.44%	5.22%	12.66%
8 - 9	843	9.81%	2.98%	5.75%	12.94%
9 - 10	703	10.31%	3.53%	6.25%	13.77%
10 - 11	589	10.90%	4.11%	6.79%	15.06%
11 - 12	525	11.51%	4.75%	7.48%	17.72%
12 - 13	487	11.51%	4.82%	7.48%	11.50%
13 - 14	430	12.21%	5.48%	8.19%	19.98%

Option close prices from CRSP used for LEAPS on the S&P 500, between 11 and 12 months to maturity.

Table 3: Summary Statistics of Nominal Financial Returns 1871-2006 from Monthly Data (Anualized except for Max and Min)

	Geometric Mean	St. Dev.	Max	Min
Stock	9.16%	14.14%	51.35%	-26.19%
Corporate Bond	4.76%	0.99%	2.01%	-0.16%
Margin Rate	4.98%	0.89%	1.5%	0.0%
Government Bond	4.64%	2.35%	1.2%	0.2%
Inflation	2.15%	3.75%	7.04%	-6.58%

Source: Shiller (2005a) except margin rate, which is money call rate from Global Financial Data, and government bond, which is Long Term US Bond Yield from Global Financial Data

Table 4: CRRA-specific Percent and Dollar Targets*

CRRA	Leveraged Targets	Unleveraged
1	160.1%	165.9%
2	88.0%	91.6%
4	45.7%	47.4%
8	23.8%	24.6%
16	12.6%	13.2%
32	6.7%	7.5%

* based on a risk free bond rate of 4.8%, margin rate 5.0%, discounted at 6.3% rate.

Table 5: Comparison of Alternative Investment Strategies Based on Optimal Investment Targets on Monthly Data

	90%/50% Strategy	% Improve. Rel. to 100% Stock	**100% Stock Strategy**	% Improve. Rel. to 100% Stock	**Dual-Target Strategy (88% & 90.6%%)**	% Improve. Rel. to 100% Stock	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock
Average % Invested in Stock (Weighted by PV Invested in Stock)	65.2%		100%		104.6%		103.5%	
Max % Inv.	90.0%		100%		200.0%		200%	
Min % Inv.	50.0%		100%		90.6%		88%	
Median	$245,826	-24.1%	$ 324,001	0.0%	$413,445	27.6%	$413,208	27.5%
Mean:	$244,989	-34.9%	$ 376,554	0.0%	$456,032	21.1%	$450,672	19.7%
Stdev	$56,434	-66.5%	$ 168,578	0.0%	$188,639	11.9%	$179,739	6.6%
Coeff of var.	23.0%	-48.5%	44.8%	0.0%	41.4%	-7.6%	39.9%	-10.9%
Min	$123,863	-5.4%	$ 130,940	0.0%	$148,736	13.6%	$149,106	13.9%
10th pct	$175,667	-13.2%	$ 202,396	0.0%	$249,476	23.3%	$251,972	24.5%
25th pct	$199,634	-22.1%	$ 256,331	0.0%	$320,554	25.1%	$319,754	24.7%
75th pct	$293,931	-39.5%	$ 485,669	0.0%	$523,783	7.8%	$520,843	7.2%
90th pct	$314,769	-52.2%	$ 659,016	0.0%	$770,292	16.9%	$757,553	15.0%
Max	$360,838	-53.5%	$ 776,184	0.0%	$877,074	13.0%	$844,517	8.8%
CRRA								
0	$244,989	-34.9%	$ 376,554	0.0%	$456,032	21.1%	$450,672	19.7%
1	$238,242	-30.6%	$ 343,237	0.0%	$419,234	22.1%	$416,344	21.3%
2	$231,201	-26.5%	$ 314,418	0.0%	$384,159	22.2%	$383,046	21.8%
4	$216,830	-19.9%	$ 270,636	0.0%	$323,017	19.4%	$323,558	19.6%
8	$191,198	-12.3%	$ 217,895	0.0%	$248,048	13.8%	$248,808	14.2%
16	$161,955	-7.3%	$ 174,774	0.0%	$197,671	13.1%	$198,227	13.4%
32	$142,826	-5.7%	$ 151,484	0.0%	$171,979	13.5%	$172,421	13.8%

Table 6: Median Age (in Years) at Phase Turning Points, Monthly Analysis*

	Single-Target	Dual-Target Strategy
Median Age When Maximum Leverage Ends	31.6 (28.4, 38.2)	31.6 (28.4, 38.2)
Median Age When All Leverage Ends	49.7 (41.2, 53.7)	49.7 (41.2, 53.7)

* 5th and 95th percentiles are given in parentheses below the median values

Table 7: 2nd Order Dominance - 100 % vs Single Target Strategy

	100% Constant	76.7% Target	% Difference	74.6% Target	% Difference
Average % Invested in Stock (Weighted by PV Invested in Stock)		100.00%		98.28%	
Median	$324,001	$361,917	11.70%	$353,346	9.06%
Mean:	$376,554	$387,336	2.86%	$376,554	0.00%
Stdev	$168,578	$130,608	-22.52%	$122,913	-27.09%
Coeff of var.	44.8%	33.7%	-24.69%	32.6%	-27.11%
Min	$130,940	$142,462	8.80%	$141,226	7.86%
10th pct	$202,396	$239,886	18.52%	$236,293	16.75%
25th pct	$256,331	$303,303	18.32%	$299,154	16.71%
75th pct	$485,669	$455,606	-6.19%	$445,073	-8.36%
90th pct	$659,016	$595,458	-9.64%	$566,370	-14.06%
Max	$776,184	$676,982	-12.78%	$646,954	-16.65%
CRRA	Certainty Equivalents:				
1	$343,237	$365,283	6.42%	$356,316	3.81%
2	$314,418	$342,725	9.00%	$335,424	6.68%
4	$270,636	$298,554	10.32%	$293,882	8.59%
8	$217,895	$235,065	7.88%	$232,606	6.75%

Table 8: Sign Test of Gross Accumulation across 93 Cohorts, Leveraged vs. Traditional Investment Strategies

	Cohorts with Accumulation Greater than 90/50% Strategy			Cohorts with Accumulation Greater than 100% Strategy		
Strategy	Number	Proportion	Significance	Number	Proportion	Significance
88%*	93	100%	infinity	84	90.32%	13.15
88/90.6%*	93	100%	infinity	84	90.32%	13.15

*cohorts that do not beat out the 100% strategy are those retiring in 1999-2007

Table 9: Analysis of Margin Calls for Different Maintenance Margin Percentages

% Market Drop from beginning of year	Maint. Margin %	# Years with Margin Calls (out of 91)	# Months with Margin Calls (out of 1597)	# Months followed by Rebound	# Months followed by Deterioration
50.00%	0.00%	0	0	0	0
40.00%	16.70%	2	2	1	1
33.30%	25.00%	4	9	5	4
30.00%	28.60%	5	13	9	4
28.60%	30.00%	6	15	11	4
23.10%	35.00%	11	30	17	13

Table 10: Prevalence of Negative Monthly Returns Among the 49,104 Cohort-Months (93x528)

Real Monthly Portfolio Return (less than or equal to)	100% Stock Strategy		Single % Target (88%) Strategy	
	# of Months	Cumm. % of Months	# of Months	Cumm. % of Months
-100.00%	0	0.0%	0	0.0%
-90.00%	0	0.0%	0	0.0%
-80.00%	0	0.0%	0	0.0%
-70.00%	0	0.0%	0	0.0%
-60.00%	0	0.0%	0	0.0%
-50.00%	0	0.0%	6	0.1%
-40.00%	0	0.0%	22	0.5%
-33.33%	0	0.0%	39	1.0%
-30.00%	0	0.0%	46	1.1%
-23.08%	44	1.1%	251	6.3%
-20.00%	88	2.2%	355	8.9%
-10.00%	922	23.0%	1919	47.9%
0.00%	19575	488.9%	19845	495.6%
Global Minimum	-26.2%		-53.0%	

Table 11: Impact of Targeting Investment Under Various Leverage Caps in 93 Age Cohorts from 1871 through 2006, Monthly Analysis

		200% Cap		300% Cap		500% Cap	
	100% Stock Strategy	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock
Wt. Av. % Invested in Stock	100%	103%		105.2%		107.3%	
Max % Inv.	100%	200.0%		300.0%		500.0%	
Min % Inv.	100%	88.0%		88.0%		88.0%	
Median	$ 324,001	$409,993	26.5%	$440,987	36.1%	$446,722	37.9%
Mean:	$ 376,554	$448,037	19.0%	$476,469	26.5%	$497,610	32.1%
Stdev	$ 168,578	$179,084	6.2%	$196,398	16.5%	$209,768	24.4%
Coeff of var.	44.8%	40.0%	-10.7%	41.2%	-7.9%	42.2%	-5.8%
Min	$ 130,940	$148,947	13.8%	$159,465	21.8%	$184,054	40.6%
10th pct	$ 202,396	$250,745	23.9%	$254,527	25.8%	$258,691	27.8%
25th pct	$ 256,331	$319,025	24.5%	$328,837	28.3%	$352,830	37.6%
75th pct	$ 485,669	$517,048	6.5%	$553,493	14.0%	$594,868	22.5%
90th pct	$ 659,016	$749,846	13.8%	$791,106	20.0%	$802,265	21.7%
Max	$ 776,184	$841,369	8.4%	$968,679	24.8%	$1,101,348	41.9%
CRRA	Certainty Equivalents						
0	$ 376,554	$448,037	19.0%	$476,469	26.5%	$497,610	32.1%
1	$ 343,237	$413,887	20.6%	$437,946	27.6%	$456,206	32.9%
2	$ 314,418	$380,881	21.1%	$401,092	27.6%	$417,426	32.8%
4	$ 270,636	$322,085	19.0%	$337,780	24.8%	$353,799	30.7%
8	$ 217,895	$248,132	13.9%	$262,276	20.4%	$282,815	29.8%
16	$ 174,774	$197,923	13.2%	$210,756	20.6%	$235,439	34.7%
32	$ 151,484	$172,229	13.7%	$184,164	21.6%	$210,574	39.0%
# of months with Wipe-outs (monthly return = -100%)	0	0		0		11	

*comparison made to 100% stock, 100% cap

Table 12: Impact of Increased Margin Rate Costs on Distribution of Retirement Wealth for Single Target Strategy

Premium Added to Margin Rate (annual)	0.0%	0.0%	1.00%	1.50%	2.00%	2.50%
	100% Stock Strategy	**Single-Target Strategy (88%)**				
Median	$324,001.09	$ 409,993	$ 393,544	$ 386,114	$ 379,438	$ 369,587
Mean:	$376,553.62	$ 448,037	$ 427,788	$ 417,655	$ 407,647	$ 397,676
Min	$130,940.21	$ 148,947	$ 140,639	$ 136,412	$ 132,161	$ 127,931
10th pct	$202,396.49	$ 250,745	$ 238,816	$ 232,532	$ 225,246	$ 218,248
Certainty Eq. (CRRA)=1	$343,237.45	$ 413,887	$ 394,823	$ 385,289	$ 375,856	$ 366,461
Certainty Eq. (CRRA)=2	$314,417.91	$ 380,881	$ 362,889	$ 353,895	$ 344,976	$ 336,101
Certainty Eq. (CRRA)=4	$270,636.45	$ 322,085	$ 305,999	$ 297,944	$ 289,913	$ 281,949
Certainty Eq. (CRRA)=8	$217,895.02	$ 248,132	$ 234,894	$ 228,215	$ 221,515	$ 214,886

Table 13: Impact of Decreased Stock Returns on Distribution of Retirement Wealth for Alternative Investment Strategies

Adjustment to Nom. Stock Return	-1.00%	-1.00%	-1.50%	-1.50%	-2.00%	-2.00%	-2.50%	-2.50%
Strategy	**100% Stock Strategy**	**Single Target (88%) Strategy**	**100% Stock Strategy**	**Single Target (88%) Strategy**	**100% Stock Strategy**	**Single Target (88%) Strategy**	**100% Stock Strategy**	**Single Target (88%) Strategy**
Median	$245,330	$299,128	$212,498	$255,266	$181,850	$218,046	$156,523	$186,921
Mean:	$281,138	$321,102	$243,326	$271,578	$210,412	$229,124	$179,503	$190,224
Min	$99,608	$107,750	$87,253	$91,765	$76,668	$78,543	$67,584	$67,511
10th pct	$145,838	$172,832	$125,519	$145,510	$107,671	$117,909	$90,820	$92,998
Certainty Eq. (CRRA)=1	$256,158	$296,689	$221,459	$250,685	$191,066	$210,894	$161,850	$173,283
Certainty Eq. (CRRA)=2	$234,469	$272,986	$202,467	$230,449	$174,271	$193,410	$146,494	$157,357
Certainty Eq. (CRRA)=4	$202,007	$231,284	$174,374	$195,354	$149,808	$163,941	$124,636	$132,357
Certainty Eq. (CRRA)=8	$164,393	$179,742	$142,832	$152,884	$123,581	$129,775	$103,092	$107,100

Table 14 - Performance of Single Target Strategy with Foreign Equities

	93 US Cohorts (1871-2007)			26 FTSE All-Shares Cohorts (1937 - 2007), Investing in UK Pounds			15 Nikkei 225 Cohorts (1956 - 2007), Investing in US$		
	100% Stock Strategy	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock	**100% Stock Strategy**	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock	**100% Stock Strategy**	**Single-Target Strategy (88%)**	% Improve. Rel. to 100% Stock
Pr. Val. Wt. Av. % Invested in Stock		109.1%			109.82%			143.98%	
Max % Inv.		200.0%			200.00%			200.00%	
Min % Inv.		88.0%			88.03%			88.05%	
Median	$ 324,001	$413,208	27.5%	£212,446.72	£265,384.94	24.9%	$116,138.61	$144,950.12	24.8%
Mean:	$ 376,554	$450,672	19.7%	£213,833.02	£260,836.33	22.0%	$135,320.19	$154,408.36	14.1%
Stdev	$ 168,578	$179,739	6.6%	£63,810.84	£60,768.61	-4.8%	$52,909.38	$54,994.20	3.9%
Coeff of var.	44.8%	39.9%	-10.9%	29.8%	23.3%	-21.9%	45.6%	37.9%	-16.7%
Min	$ 130,940	$149,106	13.9%	£96,773.17	£136,151.28	40.7%	$77,106.33	$85,286.38	10.6%
10th pct	$ 202,396	$251,972	24.5%	£127,682.65	£173,115.01	35.6%	$84,550.02	$93,567.64	10.7%
25th pct	$ 256,331	$319,754	24.7%	£173,643.02	£228,771.28	31.7%	$96,158.92	$110,410.89	14.8%
75th pct	$ 485,669	$520,843	7.2%	£251,045.30	£299,251.21	19.2%	$162,380.11	$193,464.64	19.1%
90th pct	$ 659,016	$757,553	15.0%	£311,613.57	£339,301.95	8.9%	$212,053.04	$230,994.72	8.9%
Max	$ 776,184	$844,517	8.8%	£328,764.35	£355,412.78	8.1%	$245,967.27	$255,040.60	3.7%
Cert. Eq CRRA = 1	$343,237	$413,887	20.6%	£203,722.24	£252,994.05	24.2%	$126,739.77	$145,568.81	14.9%
Cert. Eq CRRA = 2	$314,418	$380,881	21.1%	£192,563.92	£244,011.12	26.7%	$119,439.50	$137,363.51	15.0%
Cert. Eq CRRA = 4	$270,636	$322,085	19.0%	£169,329.00	£223,783.47	32.2%	$108,912.43	$124,242.98	14.1%
Cert. Eq CRRA = 8	$217,895	$248,132	13.9%	£137,112.31	£189,643.04	38.3%	$98,037.72	$109,703.83	11.9%
Cert. Eq CRRA = 16	$174,774	$197,923	13.2%	£116,293.15	£162,787.50	40.0%	$89,550.21	$99,156.88	10.7%

Table 15: Comparison of 4 Alternative Investment Strategies in 10,000 Monte Carlo Simulations (with replacement) on 1871 and 2004 Annual Returns

	90/50% Strategy	% Improve. Rel. to 100% Stock	**100% Stock Strategy**	**Dual % Target (88% & 90.6%) Strategy**	% Improve. Rel. to 100% Stock	**Single % Target (88%) Strategy**	% Improve. Rel. to 100% Stock
Max % Inv.	90.0%	-10.0%	100.0%	200.0%	0.0%	200.0%	100.0%
Min % Inv.	51.0%	-49.0%	100.0%	90.6%	3.0%	88.0%	-12.0%
Median	$278,932	-26.3%	$378,332	$450,029	19.0%	$446,726	18.1%
Mean:	$326,174	-38.7%	$532,178	$663,739	24.7%	$650,411	22.2%
Stdev	$191,403	-62.8%	$514,368	$702,013	36.5%	$671,227	30.5%
Coeff of var.	58.7%	-39.3%	96.7%	105.8%	9.4%	103.2%	6.8%
Min	$45,201	101.8%	$22,402	$20,334	-9.2%	$20,516	-8.4%
10th pct	$147,835	0.7%	$146,851	$155,867	6.1%	$156,596	6.6%
25th pct	$196,255	-13.1%	$225,887	$254,387	12.6%	$254,137	12.5%
75th pct	$398,024	-38.9%	$650,928	$812,315	24.8%	$800,106	22.9%
90th pct	$560,057	-47.5%	$1,067,594	$1,395,480	30.7%	$1,360,394	27.4%
Max	$1,956,898	-79.4%	$9,497,939	$13,749,560	44.8%	$12,650,839	33.2%
CRRA	Certainty Equivalents						
0	$326,174.42	-38.7%	$532,177.78	$663,739.19	24.7%	$650,411.42	22.2%
1	$283,354.82	-27.0%	$388,296.14	$457,967.06	17.9%	$453,778.87	16.9%
2	$248,333.07	-14.5%	$290,343.88	$321,204.64	10.6%	$320,858.02	10.5%
4	$195,353.64	14.7%	$170,385.05	$163,485.62	-4.0%	$164,437.15	-3.5%
8	$129,443.41	73.5%	$74,591.92	$65,395.00	-12.3%	$65,739.77	-11.9%
16	$80,612.89	97.0%	$40,912.32	$36,737.28	-10.2%	$36,998.09	-9.6%
32	$60,470.64	100.7%	$30,130.30	$27,321.88	-9.3%	$27,556.76	-8.5%

Table 16: Certainty Equivalent Wealth for CRRA-Specific Strategies in 10,000 Monte Carlo Simulations (with replacement)

CRRA	Certainty Equivalent Wealth	% Improve Rel. to 90/50	% Improve. Rel. to 100% Stock
2 (88.0%)	$320,858	29.2%	10.5%
4 (45.7%)	$204,174	4.5%	19.8%
8 (23.8%)	$143,546	10.9%	92.4%
16 (12.6%)	$104,658	29.8%	155.8%
32 (6.7%)	$79,229	31.0%	163.0%

Table 17 - Temporal Vs Asset Diversification in S&P 500 Components 1986-2006

	Time	Assets	
	1/21th of Portfolio in 1 Random Stock Each Year*	Full Portfolio in 21 Random Stocks One Year*	Full Portfolio in Equal-Weight S&P 500 One Year*
Trials	1418	1441	21
Mean	5.51%	5.51%	5.50%
Std. Deviation	4.88%	9.74%	8.40%
10th Percentile	5.04%	4.40%	4.82%
25th Percentile	5.25%	5.09%	5.27%
50th Percentile	5.48%	5.55%	5.57%
75th Percentile	5.72%	5.92%	5.78%
90th Percentile	5.97%	6.34%	6.09%

*In the first column, an investor is modelled as each year choosing one new random stock from the S&P 500 and investing 1/21 of his portfolio in this, with the balance invested in T-Bills. In the second two columns, the investor chooses one random year in which to invest in, respectively, either 21 randomly chosen S&P 500 stocks or an equal weight portfolio. In the other years he invests all his money in T-Bills.