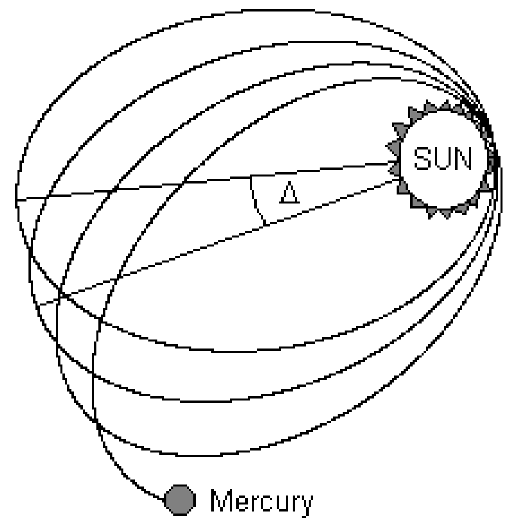
Lisa Larrimore – Physics 111

Mercury and the Relativistic Correction for Precession of the Apsides

We have seen that the orbits of the planets around the sun can be found, to a good approximation, by only considering a two-body system (a planet and the sun) interacting through a central force. However, all the other planets cause slight variations in the predicted path of an individual planet.

Astronomers have been able to calculate the expected precession of Mercury's apsides (due to the forces of other planets) to be 531 arcseconds per century, and they have observed that the actual precession is 574 arcseconds per century. This leaves a difference of 43" that could not be explained by uncertainties in calculation or measurement and which was noticed as early as 1845.



Before Einstein's theory of relativity, there were three unsatisfactory theories to explain this phenomenon: a retarding force due to a dust cloud around the sun, a new planet between Mercury and the sun, and an exponent slightly different from -2 in the gravitational force law. Then Einstein modified the force law by introducing a component that varies as $1/r^4$.

We will use this correction to calculate the angle Δ through which the apsides of Mercury shift with each revolution, ignoring the effects of other planets, as shown here in the exaggerated figure.

Using the substitution $r = 1/u$, we can write the equation of motion given in Eq. 8.21 as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{1}{u^2}F(1/u) \ ,$$

where the gravitational force with the relativistic correction is given by

$$F = -\frac{GmM}{r^2} - \frac{3GM \cdot l^2}{mc^2 r^4} = -GmMu^2 - \frac{3GM \cdot l^2}{mc^2}u^4 \ .$$

Making the substitutions

$$\frac{1}{\alpha} \equiv \frac{Gm^2M}{l^2} \ , \qquad \delta \equiv \frac{3GM}{c^2}$$

we have

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \ . \qquad (1)$$

Though we cannot solve this differential equation exactly, we can use a *successive approximation procedure* to get an approximate analytic solution. First we ignore the relativistic term δu^2, giving us the familiar solution

$$u_1 = \frac{1}{\alpha}(1 + \varepsilon\cos\theta) \ .$$

Substituting u_1 into the right side of Eq. 1, we have

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{\alpha}+\frac{\delta}{\alpha^2}\left[1+2\varepsilon\cos\theta+\frac{\varepsilon^2}{2}(1+\cos 2\theta)\right]. \qquad (2)$$

Substituting u_1 into the left side of Eq. 1 only gives the $1/\alpha$ term, and it turns out that substituting

$$u_p=\frac{\delta}{\alpha^2}\left[\left(1+\frac{\varepsilon^2}{2}\right)+\varepsilon\theta\sin\theta-\frac{\varepsilon^2}{6}\cos 2\theta\right]$$

into the left side of Eq. 1 generates the remainder of Eq. 2, as you can verify with a little calculus and algebra.

If we stop the successive approximation procedure at this point, we have

$$u\cong u_2=u_1+u_p=\left[\frac{1}{\alpha}(1+\varepsilon\cos\theta)+\frac{\delta\varepsilon}{\alpha^2}\theta\sin\theta\right]+\left[\frac{\delta}{\alpha^2}\left(1+\frac{\varepsilon^2}{2}\right)-\frac{\delta\varepsilon^2}{6\alpha^2}\cos 2\theta\right].$$

The second bracketed term contains a constant and a small *periodic* disturbance, whereas the $\theta\sin\theta$ term produces *secular* effects, so

$$u_{secular}=\frac{1}{\alpha}\left[1+\varepsilon\cos\theta+\frac{\delta\varepsilon}{\alpha}\theta\sin\theta\right]\cong\frac{1}{\alpha}\left[1+\varepsilon\cos\left(\theta-\frac{\delta}{\alpha}\theta\right)\right], \qquad (3)$$

where we have used the approximations

$$\cos\frac{\delta}{\alpha}\theta\cong 1, \qquad \sin\frac{\delta}{\alpha}\theta\cong\frac{\delta}{\alpha}\theta .$$

In choosing u_1, we chose to measure θ from the perihelion distance ($r_{\min}=u_{\max}$) at $t=0$, so the next perihelion will occur when the argument of the cosine term in Eq. 3 is 2π. Thus,

$$\theta=\frac{2\pi}{1-(\delta/a)}\cong 2\pi\left(1+\frac{\delta}{\alpha}\right).$$

So the relativistic term in the force law causes a displacement of perihelion (and thus the aphelion) in each revolution by

$$\Delta\cong\frac{2\pi\delta}{\alpha}=6\pi\left(\frac{GmM}{cl}\right)^2\cong\frac{6\pi GM}{ac^2(1-\varepsilon^2)},$$

making the approximation $\mu\cong m$ in the last step. Since Mercury is the planet with the smallest a and largest ε, this effect is most noticeable. To calculate this, we put in lots of constants and get

$$\Delta=\frac{6\pi(\mathrm{rad/rev})\left(6.6726\cdot 10^{-11}\cdot\mathrm{m^3/s^2\ kg}\right)\left(333480\cdot 5.976\cdot 10^{24}\cdot\mathrm{kg}\right)}{\left(0.3871\cdot 1.495\cdot 10^{11}\cdot\mathrm{m}\right)\left(3\cdot 10^{8}\ \mathrm{m/s}\right)^2\left(1-0.2056^2\right)}=5.0249\cdot 10^{-7}\cdot\mathrm{rad/rev}$$
$$=\left(5.0249\cdot 10^{-7}\cdot\mathrm{rad/rev}\right)\left(\frac{180\cdot\mathrm{deg}}{\pi\cdot\mathrm{rad}}\right)\left(\frac{3600\cdot\mathrm{arcsec}}{\mathrm{deg}}\right)\left(\frac{1\cdot\mathrm{rev}}{0.2408\cdot\mathrm{yr}}\right)\left(\frac{100\cdot\mathrm{yr}}{1\cdot\mathrm{century}}\right)\cong 43\cdot\mathrm{arcsec/century},$$

which was the unexplained precession of Mercury’s apsides!