Floating Bodies of Equilibrium

Franz Wegner, Institut für Theoretische Physik
Ruprecht-Karls-Universität Heidelberg
Philosophenweg 19, D-69121 Heidelberg
Email: wegner@tphys.uni-heidelberg.de

Introduction

Yacov Kantor in his series of Physics questions posed the problem 06/98 (http://star.tau.ac.il/QUIZ/):

Consider a long rod of circular cross-section; it will, obviously, float in any position without tending to rotate. (Of course, the axis of the log is assumed to be parallel to the water surface.)

(a) Are there any additional cross-section shapes such that the log will float in any position if the density of the floating body is 0.5 g/cm^3?

(b) ... (c) ...

Here I will argue that the answer to question (a) is affirmative. More precisely, if $s(\phi)$ obeys

$$s(\phi+\pi) = -s(\phi), \tag{1}$$

and the following two integrals vanish

$$\int_0^\pi s(\phi)\cos\phi\, d\phi = 0, \quad \int_0^\pi s(\phi)\sin\phi\, d\phi = 0 \tag{2}$$

then a log with the boundary given in parametric representation

$$x(\phi) = x_M(\phi) + l\cos\phi, \qquad y(\phi) = y_M(\phi) + l\sin\phi, \tag{3}$$

$$x_M(\phi) = \int_0^\phi s(\phi)\cos\phi\, d\phi, \qquad y_M(\phi) = \int_0^\phi s(\phi)\sin\phi\, d\phi \tag{4}$$

with sufficiently large l so that the log is convex will float in any orientation. Convexity is fulfiiled, if $\frac{d}{d\phi}(\frac{dy}{d\phi}/\frac{dx}{d\phi})$ has the same sign for all ϕ, which is the case, if $l^2 + s^2(\phi) > l\frac{ds(\phi)}{d\phi}$ for all ϕ.

General Considerations

In order to derive this result I start with some considerations for general densities ρ. I have to introduce some notation. I denote the cross-section area of the log by A, the cross-section above the water is A_1, that below the water A_2, then

$$A_1 = (1-\rho)A, \quad A_2 = \rho A \tag{5}$$

according to Archimedes (I assume the density of water to equal 1 without any dimension). Moreover I denote by m the total mass of the log, by $m_{1,2}$ the masses above/below the water-line, by $C_{1,2}$ the center of mass above/below the water-line, by $h_{1,2}$ the distance of $C_{1,2}$ from the water-line, and by $\hat{L}$ the length of the log.

Then the potential energy of the system is

$$V = m_1 g h_1 + (m - m_2) g h_2 = \rho A_1 \hat{L} h_1 g + (A_2 \hat{L} - A_2 \rho \hat{L}) g h_2 = \rho(1-\rho) A \hat{L} g (h_1 + h_2) \tag{6}$$

Thus the difference in height between the two centers of mass, $h_1 + h_2$, has to be independent of the orientation of the log. This does not imply, that h_1 and h_2 are separately constant. Moreover the line $C_1 C_2$ connecting the two centers of mass has to be perpendicular to the water-level. In the following I will consider only convex cross-sections. Thus the water-level crosses the circumference of the log twice at points L and R. I denote the distance between these two points by $2l$.

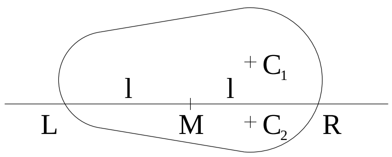

Figure 1

Let us consider now what happens if we rotate the log by an infinitesimal angle $\delta\phi$. Since the area above and below the water has to be conserved the rotation is around the midpoint M of the line LR. Then on the right the area $l^2 \delta\phi/2$ rises above the water and on the left the same area disappears below the water-level. Let us consider the shift of the centers of mass in horizontal direction. There are two changes: Due to the rotation C_1 moves to the left by $\delta\phi h_1$ and C_2 to the right by $\delta\phi h_2$. The appearance of the area $l^2\delta\phi/2$ on the right and the disappearance of the same area on the left moves the center C_1 of mass by $\delta\phi(\int_0^l l^2 dl - \int_{-l}^0 l^2 dl)/A_1 = \frac{2}{3}\delta\phi l^3/A_1$ to the right. Similarly the center of mass below the water level C_2 moves to the left by $\frac{2}{3}\delta\phi l^3/A_2$.

In total the shift to the right of C_1 is $(-h_1 + \frac{2}{3}l^3/A_1)\delta\phi$ and that of C_2 $(h_2 - \frac{2}{3}l^3/A_2)\delta\phi$. Both have to be equal since otherwise a torque would be created. Thus we have

$$\frac{2}{3} l^3 (\frac{1}{A_1} + \frac{1}{A_2}) = h_1 + h_2. \tag{7}$$

Since the distance $h_1 + h_2$ has to be constant and also A_1 and A_2 are constant, we obtain the important result, that also l is constant. If (7) is not fulfilled, but the left hand side is larger then the right hand side, then the log is in stable equilibrium, if the left hand side is less then the right hand side then the equilibrium is unstable. But here we are interested in an indifferent equilibrium, where equality (7) holds.

When we continue to rotate the log then in general the mid-point M between L and R will move. It can move only along the direction of the line LR. We consider now the loci of the points M as a function of ϕ. In order to do this we keep however the orientation of the log fixed and rotate the direction of the gravitational force and the orientation of the watersurface. Since M can move only in direction of the line LR whose slope is now given by $\tan\phi$ we can parametrize the loci of M by $x_M(\phi)$ and $y_M(\phi)$ as given in (4), where $s(\phi)d\phi$ is the shift of M along LR as we rotate by $d\phi$. It is now obvious that the solution of the problem can be parametrized by $x(\phi)$, $y(\phi)$ as given above. It must be noted however, that the boundary of the log is not only given by eq. (3), but it is also obtained by reversing the sign of l, that is not only by considering the points R but also the points L

$$x(\phi) = x_M(\phi) - l\cos\phi, \quad y(\phi) = y_M(\phi) - l\sin\phi. \tag{8}$$

A priori it is not clear, whether this expression describes the same boundary, resp., what are the conditions on $s(\phi)$ so that both descriptions coincide. Of course one expects different arguments ϕ in both expressions in order to obtain the same point on the boundary. It is also obvious, that for constant $s(\phi)$ we obtain a circular cross-section. The reader may verify, that eq. (7) holds in this case.

The case $\rho = \frac{1}{2}$

In the special case $\rho = \frac{1}{2}$ one has $A_1 = A_2 = A/2$, with the implication that for ϕ and $\phi+\pi$ one has the same line LR separating the part of the log above and that below the water-level, only L and R are exchanged, but M is the same

$$x_M(\phi) = x_M(\phi+\pi), \quad y_M(\phi) = y_M(\phi+\pi). \tag{9}$$

If we differentiate these equations with respect to ϕ we obtain the condition (1) and taking it for $\pi = 0$ yields (2). These conditions are equivalent to (9).

As an example we consider

$$s(\phi) = a\cos(3\phi) \tag{10}$$

which yields

$$x_M(\phi) = \frac{a}{4}\sin(2\phi) + \frac{a}{8}\sin(4\phi), \quad y_M(\phi) = \frac{a}{4}\cos(2\phi) - \frac{a}{8}\cos(4\phi) - \frac{a}{8}, \tag{11}$$

a curve which has kinks at $(\pm 3\sqrt{3}a/16,\ a/16)$, $(0,\ -a/2)$. For $l > 3a$ the circumference is convex, but it might be that it is sufficient to have $l > a/2$, so that the circumference lies completely outside the loci of M.

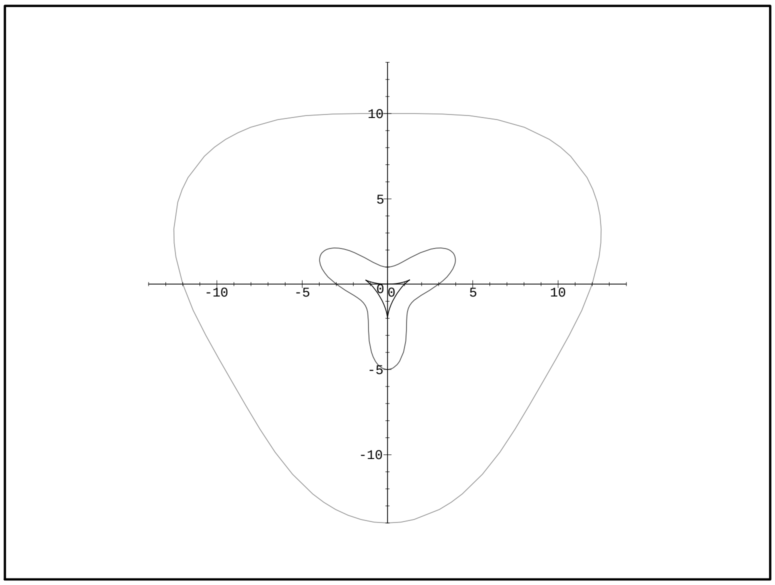

Figure 2. This figure shows the loci of M for $a = 4$ (innermost curve) and the boundaries for $l = 3$ and $l = 12$ (outermost curve). A log with the cross-section of the outermost curve and $\rho = 1/2$ will flow in water in any position without tendency to rotate. The water-line will be tangent to the innermost curve.

We finally show, that (7) is fulfilled. For this purpose we integrate $A_1 h_1$ and $A_2 h_2$. We assume without loss of generality, that the water-level obeys $\phi = 0$.

$$A_1 h_1 = -\frac{1}{2}\int_0^{\pi} y^2 \frac{dx}{d\phi} d\phi \tag{12}$$

$$A_2 h_2 = \frac{1}{2}\int_{\pi}^{2\pi} y^2 \frac{dx}{d\phi} d\phi \tag{13}$$

The minus-sign in the first expression is introduced since x decreases mainly with ϕ. We have now to insert the expressions for y, (3) and

$$\frac{dx}{d\phi} = s(\phi)\cos\phi - l\sin\phi \tag{14}$$

The integrand can now be written

$$\begin{aligned} y^2 \frac{dx}{d\phi} &= (-l^3 \sin^3\phi - y_M^2 l \sin\phi + 2 y_M l \sin\phi \cos\phi\, s(\phi) \\ &+ (l^2 \sin^2\phi \cos\phi\, s(\phi) - 2 y_M l^2 \sin^2\phi + y_M^2 \cos\phi\, s(\phi)) \end{aligned} \tag{15}$$

Under the replacement $\phi \to \phi + \pi$ function $s(\phi)$ changes sign, whereas $y_M(\phi)$ is invariant. Therefore the first parenthesis in (15) changes sign the second one does not. The contribution from the second parenthesis cancels in

$$\begin{aligned} A_1 h_1 + A_2 h_2 &= \int_0^\pi (l^3 \sin^3 \phi + y_M^2 l \sin \phi - 2 y_M l \sin \phi \cos \phi \, s(\phi)) d\phi \\ &= \frac{4}{3} l^3 - y_M^2 l \cos \phi \big|_0^\pi \qquad (16) \end{aligned}$$

Since $y_M(0) = y_M(\pi) = 0$ the last term vanishes and (7) is fulfilled.

In the example given above (10, 11) h_1 and h_2 are not constant, only their sum is constant. For $\phi = 0$ one obtains

$$A_1 = A_2 = (\frac{1}{2} l^2 - \frac{1}{32} a^2)\pi \qquad (17)$$

$$A_1 h_1 = \frac{2}{3} l^3 - \frac{1}{16} l^2 a\pi + \frac{1}{256} a^3 \pi, \qquad (18)$$

$$A_2 h_2 = \frac{2}{3} l^3 + \frac{1}{16} l^2 a\pi - \frac{1}{256} a^3 \pi \qquad (19)$$

Since under rotation by π the heights h_1 and h_2 exchange their roles it is evident that h_1 and h_2 are not separately constant.