The first part is simple. Given Ψ_0, one can substitute in Schrödinger equation

$$-\Psi'' + V(x)\Psi = E\Psi \tag{1}$$

($V(x)$ and E are measured in units of $\frac{\hbar^2}{2m}$). Therefore $V(x) = E_0 + \frac{\Psi_0''}{\Psi_0}$. The value of E_0 is just a matter of gauge.

The answer to the second part. The idea is to start from a known function f which is the ratio of Ψ_1 and Ψ_0.

$$\Psi_1 = f\Psi_0 \tag{2}$$

Using the definition of f and Schrödinger equation, we can derive expressions for $V(x)$, $\Psi_0(x)$ and $\Psi_1(x)$. There are some constraints that we should impose on f. For example f should have only one zero (This is because Ψ_0 has no zeros and Ψ_1 has one zero).

The solution: Let us impose Schrödinger equation on Ψ_0 and Ψ_1

$$-\Psi_0'' + V(x)\Psi_0 = E_0\Psi_0 \tag{3}$$

$$-(f\Psi_0)'' + V(x)f\Psi_0 = E_1 f\Psi_0 \tag{4}$$

Substituting (3) in (4) we obtain a differential equation for Ψ_0

$$\Psi_0' = -\frac{f}{2f'}\left(\frac{f''}{f} + \Delta E\right)\Psi_0, \tag{5}$$

where ΔE is $E_1 - E_0$. The solution of (5) is

$$\Psi_0 = \frac{1}{\sqrt{f'}} e^{-\Delta E \int^x \frac{f}{2f'}} \tag{6}$$

Thus, given $f(x)$ an expression for Ψ_0 is at hand. Ψ_1 is given by (2) and the potential $V(x)$ is derived using the solution of the first part the question.

Example: Let $f(x) = x^n$. Using the above derivation we obtain $\Psi_0 = \frac{1}{\sqrt{nx^{n-1}}} e^{-\frac{\Delta E}{4n}x^2}$, $\Psi_1 = \frac{x^n}{\sqrt{nx^{n-1}}} e^{-\frac{\Delta E}{4n}x^2}$ and $V(x) = \frac{\Psi_0''}{\Psi_0}$. (I used $E_0 = 0$ gauge). In the case of $n = 1$ we get the harmonic oscillator solution. However $n \neq 1$ corresponds to non-trivial potentials.