Charge distribution with spherically symmetric potential

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1 Problem statement

Given is a general charge distribution within a sphere of radius R. It is known that outside the sphere, the potential is spherically symmetric and given by the expression A/r. What can then be said about the charge distribution?

2 Solution

First some notations and conventions. Without loss of generality we can assume that the radius of the sphere is 1, by properly selecting our units. Let the sphere be denoted by the symbol S, and its surface by ∂S. Points within the sphere are denoted by $\overline{P}$, and points on the surface by $\overline{Q}$. The charge distribution is $\rho(\overline{P})$, and the potential on the surface is $V(\overline{Q})$. Then in general,

$$V(\overline{Q}) = \int_S \rho(\overline{P}) \frac{1}{|\overline{P} - \overline{Q}|} d\overline{P}.$$

A potential of the form A/r can be generated by a uniform charge distribution within S, ρ_0. By subtracting ρ_0 from ρ, the problem is reduced to finding the ρ for which the outer potential is zero. Due to Gauss' law, however, if the potential can be shown to be zero on ∂S only, it is zero everywhere outside S. The problem can thus be restated as finding a ρ such that $V(\overline{Q}) = 0$.

Let us first expand $V(\overline{Q})$ into spherical harmonics $Y_l^m(\theta, \phi)$, where θ is the polar angle and ϕ the azimuth. On a spherical surface these Y functions are known to form a complete orthonormal set for l and m integers and $0 \leq m \leq l$. We thus have

$$\begin{aligned} V(\overline{Q}) &= \sum_{l,m} v_{l,m} Y_l^m(\overline{Q}), && (1) \\ v_{l,m} &= \int_{\partial S} V(\overline{Q}) Y_l^m(\overline{Q}) d\overline{Q}. && (2) \end{aligned}$$

We can rewrite the expression for the expansion coefficients in the following way. First substitute the expression for V, and then switch integrals:

$$\begin{aligned} v_{l,m} &= \int_{\partial S} d\overline{Q} \left[\int_S d\overline{P} \frac{1}{|\overline{P} - \overline{Q}|} \rho(\overline{P}) \right] Y_l^m(\overline{Q}) && (3) \\ &= \int_S d\overline{P} \left[\int_{\partial S} d\overline{Q} \frac{1}{|\overline{P} - \overline{Q}|} Y_l^m(\overline{Q}) \right] \rho(\overline{P}) && (4) \\ &= \int_S d\overline{P} W_{l,m}(\overline{P}) \rho(\overline{P}), && (5) \end{aligned}$$

where $W_{l,m}(\overline{P})$ is defined to be the inner integral. By inspection one can see that $W_{l,m}(\overline{P})$ is formally equal to the potential in $\overline{P}$ caused by a surface charge distribution $Y_l^m(\overline{Q})$ on ∂S.

The requirement that $V(\overline{Q})$ vanishes on ∂S translates to the requirement that every $v_{l,m}$ be zero, or, alternatively,

$$\int_S d\overline{P} W_{l,m}(\overline{P}) \rho(\overline{P}) = 0,$$

for every l, m.

Now, again due to the completeness of the Y functions, *any* surface charge distribution $\hat{\rho}$ on ∂S can be expanded into spherical harmonics (hats are added to distinguish from the previous charge distributions and potentials):

$$\hat{\rho}(\overline{Q}) = \sum_{i,j} d_{i,j} Y_i^j(\overline{Q}),$$

and the potential caused by this inside the sphere is

$$\hat{V}(\overline{P}) = \sum_{i,j} d_{i,j} W_{i,j}(\overline{P}).$$

The requirement that V vanishes on ∂S now becomes:

$$\int_S d\overline{P} \hat{V}(\overline{P}) \rho(\overline{P}) = 0,$$

for any surface charge distribution $\hat{\rho}(\overline{Q})$. But $\hat{V}$ is, by definition, a harmonic function, and can be expanded into the (non-orthogonal) set of functions $Y_i^j(\theta, \phi) r^i$. So we must have

$$\int_S d\overline{P} Y_l^m(\theta, \phi) r^l \rho(\overline{P}) = 0,$$

for any l, m $(0 \leq m \leq l)$.

The charge distribution $\rho(\overline{P})$ is not harmonic and must be expanded into a more general set $Y_i^j(\theta, \phi) f_k(r)$. The functions f_k must be orthogonal over the interval $[0, 1]$, with respect to the weight function r^2 (the r-dependent factor of the volume element in spherical coordinates). Functions satisfying this requirement are the Jacobi polynomials $f_k(r) = P_k^{(0,2)}(2r - 1)$. One can arrive at these starting from orthogonalising the set r^k, using a Gramm-Schmidt procedure.

If, however, ρ is to cause a zero potential on ∂S, then not all functions of this complete set are allowed for the representation of ρ. In fact, we must have

$$\int_S d\overline{P} Y_l^m(\theta, \phi) r^l Y_i^j f_k(r) = 0.$$

The Y-functions are already orthonormal, so the only requirement left over is that

$$\int_0^1 dr r^2 r^i f_k(r) = 0,$$

when $l = i$ and $m = j$. It is a property of the Jacobian polynomials that

$$\int_0^1 dr r^{i+2} P_k^{(0,2)}(2r - 1) = 0,$$

only when $k > i$. Using integral 7.391.3 in Gradshteyn and Ryzhik this integral evaluates to

$$\frac{(i+2)!i!}{k!(i-k)!(i+k+3)!} = \frac{i(i-1)\cdots(i-k+1)}{k!(i+k+3)(i+k+2)\cdots(i+3)},$$

which indeed is zero only for $i = 0, 1, \ldots, k-1$.

This gives thus our final result: The charge distribution ρ producing a A/r potential on ∂S must be a linear combination of the functions

$$Y_i^j(\theta, \phi) P_k^{(0,2)}(2r - 1),$$

with $0 \leq j \leq i < k$, and the constant function 1, both functions restricted to the unit sphere $r \leq 1$.