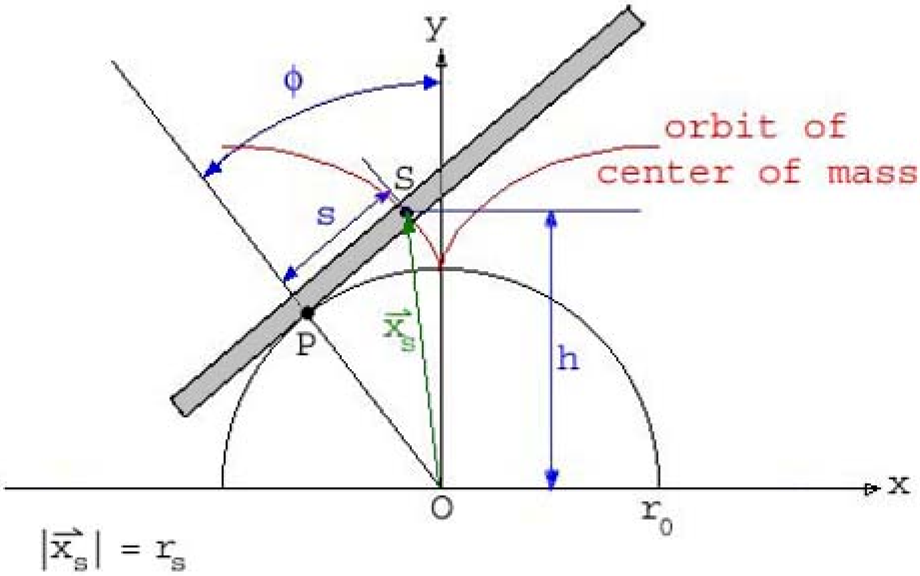
Abstract

On the following pages you will find a solution to the problem posted in march 03 at Yacov Cantor's Physics Quiz (see: http://star.tau.ac.il/QUIZ/)
posted by: Matthias Punk (matze.p@gmx.net)

"Plank on a Log" In this problem, the question was to calculate the frequency of an oscillating plank on a log. Here is a litle picture of the situation:



In the following calculation 'm' is the mass of the plank and 'J' is the component of the momentum of inertia of the plank with respect to the center of mass (of the plank) and the given rotation axis.

The Lagrangian of this problem is given by

$$L = \frac{m}{2}(\dot{\vec{x_s}})^2 + \frac{1}{2}J\omega^2 - mgh$$

Using polar coordinates and the condition

$$s = R\phi$$

(with $R = (r_0 + \frac{d}{2})$ where d is the thickness of the plank)you get

$$h = R\,cos(\phi) + s\,sin(\phi) = R(cos(\phi) + \phi\,sin(\phi))$$

and

$$\vec{x_s} = R\begin{pmatrix} \phi\,cos(\phi) - sin(\phi) \\ cos(\phi) + \phi\,sin(\phi) \end{pmatrix}$$

$$\dot{\vec{x}_s} = R\begin{pmatrix} -\phi\,\dot{\phi}\,\sin(\phi) \\ \phi\,\dot{\phi}\,\cos(\phi) \end{pmatrix}$$

$$(\dot{\vec{x}_s})^2 = R^2\,\phi^2\,\dot{\phi}^2$$

The Lagrangian is now given by

$$L = \dot{\phi}^2\,(\frac{m}{2}R^2\phi^2 + \frac{1}{2}J) - mgR(cos(\phi) + \phi\,sin(\phi))$$

In this case the Hamiltonian is conserved and equals the total energy E of the system. With this you find

$$(\frac{d\phi}{dt})^2 = \frac{E - mgR(cos(\phi) + \phi sin(\phi))}{\frac{m}{2}R^2\phi^2 + \frac{1}{2}J}$$

Knowing this you can easily derive the time period of the oscillating plank by

$$T = \frac{1}{\nu} = 2\int_{\phi_{min}}^{\phi_{max}} \sqrt{\frac{\frac{m}{2}R^2\phi^2 + \frac{1}{2}J}{E - mgR(cos(\phi) + \phi sin(\phi))}}\,d\phi$$

with E given by your initial conditions.

For example, if

$$\phi(t=0) = \phi_0$$

$$\dot{\phi}(t=0) = 0$$

then

$$\phi_{min} = -\phi_{max} = \phi_0$$

and

$$E = E_{pot_0} = mgR(cos(\phi_0) + \phi_0 sin(\phi_0))$$

If you want to have an analytical result for small angles, probably the easiest way is to write down the Euler-Lagrange-Equation

$$\ddot{\phi}\,(mR^2\phi^2 + J) + \dot{\phi}^2 mR^2\phi + mgR\phi\,cos(\phi) = 0$$

and expand it to the first order for small angles. If you do this you will find

$$\ddot{\phi}\,J + mgR\phi = 0$$

For small angles ν is now given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{mgR}{J}}$$