A brick on three sticks

1 Problem analysis

Let us first consider a particular case with two sticks at each end of the brick and the last one in the middle (see Fig. 1). One could consider that the brick is at equilibrium relying on the middle stick only, with no force applied to the two other ones. An alternative possibility is that the two extreme sticks hold all the weight and that no force is exterted onto the middle stick. Both views neither violate the intertia theorem (the sum of the forces is zero) nor contradict the moments equation (their sum is also zero). To put it into equations, the forces and their moments give us two equations for three unknown quantities:

$$P = F_1 + F_2 + F_3 \tag{1}$$

$$0 = x_1 F_1 + x_2 F_2 + x_3 F_3 \tag{2}$$

The two cases we consider cannot actually be observed: the brick is slightly deformable and bends under its own weight. Figure 2 shows the effect of this bending, which discards the solutions we had thought of. Another possibility is the compression of the sticks under the weight of the brick (Figure 3). Whether one or the other possibility depends on the deformability of the brick and of the sticks.

2 Solution to the stick compression problem

We first consider compressible sticks and a rigid brick. We carry out the linear compression approximation by stating that the of the top of the stick number i has a vertical coordinate

$$y_i = -kF_i, \tag{3}$$

where $y_i = 0$ for an uncompressed stick.

Since the bar is rigid the tops of the sticks —in contact with the bar within the approximation of a small slope— are aligned, so that

$$(y_1 - y_2)(x_1 - x_3) = (x_1 - x_2)(y_1 - y_3), \tag{4}$$

which leads to

$$F_1(x_3 - x_2) + F_2(x_1 - x_3) + F_3(x_2 - x_1) = 0. \tag{5}$$

Together with equations (1,2), equation (5) allows us to retrieve the three forces:

$$F_i = \frac{x_j(x_k - x_i) + x_k(x_j - x_i)}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3} \cdot \frac{P}{2} \tag{6}$$

3 Solution to the bending problem

We investigate here the solution of rigid sticks and a deformable brick. The flexion of the brick is given by the equation

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = kf, \tag{7}$$

where f is the force per length unit, x the horizontal coordinate, $y(x)$ the flexion of the brick and k a rigidity coefficient. In the present case, the forces are the weight and the reactions of the sticks, so that

$$f = \frac{P}{L} + F_1\delta(x - x_1) + F_2\delta(x - x_2) + F_3\delta(x - x_3). \tag{8}$$

It is straightforward to derive y:

$$y'''(x)/k = F_1\Pi^{(0)}(x - x_1) + F_2\Pi^{(0)}(x - x_2) + F_3\Pi^{(0)}(x - x_3) + \frac{P}{L}x + A, \tag{9}$$

$$y''(x)/k = F_1\Pi^{(1)}(x - x_1) + F_2\Pi^{(1)}(x - x_2) + F_3\Pi^{(1)}(x - x_3) + \frac{P}{2L}x^2 + Ax + B, \tag{10}$$

$$y'(x)/k = \frac{F_1}{2}\Pi^{(2)}(x - x_1) + \frac{F_2}{2}\Pi^{(2)}(x - x_2) + \frac{F_3}{2}\Pi^{(2)}(x - x_3) + \frac{P}{6L}x^3 + \frac{A}{2}x^2 + Bx + C, \tag{11}$$

$$y(x)/k = \frac{F_1}{6}\Pi^{(3)}(x - x_1) + \frac{F_2}{6}\Pi^{(3)}(x - x_2) + \frac{F_3}{6}\Pi^{(3)}(x - x_3) + \frac{P}{24L}x^4 + \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D, \tag{12}$$

with

$$\Pi^{(n)}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^n & \text{otherwise} \end{cases} \tag{13}$$

The solution must fulfill that the flexion is zero on each stick, and that the moments and torques exterted on both ends of the brick are zero. Thence:

$$y(x_1) = y(x_2) = y(x_3) = 0, \tag{14}$$

$$y''(-L/2) = y''(+L/2) = 0, \tag{15}$$

$$y'''(-L/2) = y'''(+L/2) = 0. \tag{16}$$

It is a set of linear equations from which A, B, C, D, F_1, F_2 and F_3 can be obtained:

$$F_1 = \frac{P}{16L} \frac{(x_1 - x_3)Q + R + 16Lx_2(x_1 - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \tag{17}$$

$$F_2 = \frac{P}{16L} \frac{(x_1 - x_3)Q + R}{(x_2 - x_1)(x_2 - x_3)}, \tag{18}$$

$$F_3 = \frac{P}{16L} \frac{(x_1 - x_3)Q + R - 16Lx_1(x_3 - x_2)}{(x_3 - x_2)(x_3 - x_1)}, \tag{19}$$

with

$$Q = 2(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 + 3L^2), \tag{20}$$

$$R = [4(x_1^2 + x_2^2) + 4x_2(x_1 - 3x_3)]L. \tag{21}$$

We did not use the sum of the forces and the sum of their moments because they are contained in the equations. It can also be checked that the solution meets Eqs.(1,2). Another striking point is that the rigidity of the brick does not appear, so that the result is material-independent.

We have to check whether the forces are directed upwards as they ought to be. Two possibilities may occur according to the location of the sticks, as shown in Fig. 4: if all forces are found to be positive, then the brick relies on the three sticks. If one of the forces is negative, the solution is not consistent, so we conclude that the corresponding stick does not support the brick. A few examples are given in Table 1.

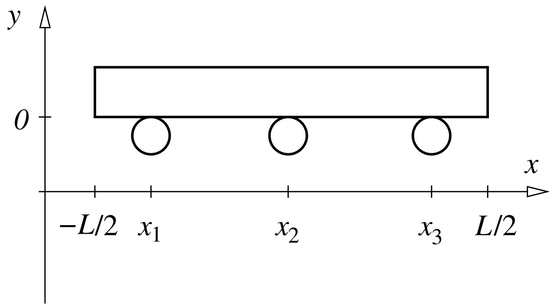

Figure 1: Slice view of three sticks supporting a brick.

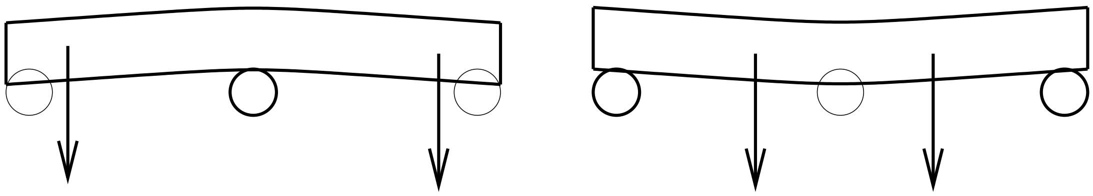

Figure 2: Deformation of the brick under its weight: impossibility for neither the middle stick nor the two extreme ones to support all the weight of the brick.

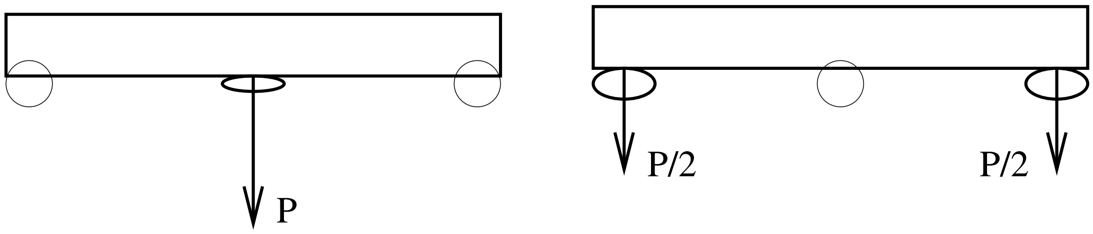

Figure 3: Deformation of the sticks under the weight of the brick: neither the two end sticks nor the middle one can support the brick alone.

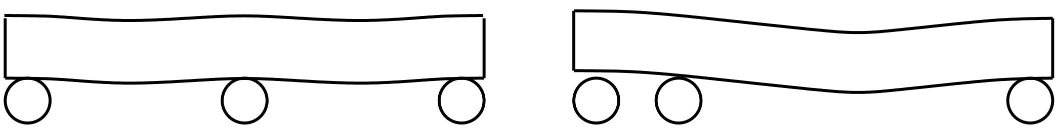

Figure 4: The brick is supported by either two or three sticks.

The middle is stick moved						the sticks at both ends are moved					
x_1/L	x_2/L	x_3/L	$F_1/\|P\|$	$F_2/\|P\|$	$F_3/\|P\|$	x_1/L	x_2/L	x_3/L	$F_1/\|P\|$	$F_2/\|P\|$	$F_3/\|P\|$
-0.5	-0.4	0.5	–	5/9	4/9	-0.5	0.0	0.5	3/16	5/8	3/16
-0.5	-0.3	0.5	–	5/8	3/8	-0.4	0.0	0.4	43/160	37/80	43/160
-0.5	-0.2	0.5	–	5/7	2/7	-0.3	0.0	0.3	33/80	7/40	33/80
-0.5	-0.1	0.5	27/240	31/48	29/120	-0.2	0.0	0.2	1/2	–	1/2
-0.5	0.0	0.5	3/16	5/8	3/16	-0.1	0.0	0.1	1/2	–	1/2

Table 1: Forces applied by the sticks onto the brick: dependence on the location of the sticks.