Line of Charges

Let N "charges" at the positions $x_i, i = 1, \ldots, N$ interact via the potential

$$V(x_1, \ldots, x_N) = \sum_{i=1}^{N} \frac{x_i^2}{2} - \sum_{i<j} \ln |x_i - x_j|. \tag{1}$$

The stationary points of $V(x_1, \ldots, x_N)$ are determined by the equation

$$\frac{\partial V(x_1, \ldots, x_N)}{\partial x_k} = x_k - \sum_{i \neq k} \frac{1}{x_k - x_i} = 0. \tag{2}$$

On the other hand the Hermite polynomial $H_N(x)$ satisfies the differential equation

$$H_N(x)'' - 2xH_N(x)' + 2NH_N(x) = 0. \tag{3}$$

Since $H_N(x)$ has N distinct real roots $x_i, i = 1, \ldots, N$ we can write

$$H_N(x) = H_N(x - x_1) \cdot \cdots \cdot (x - x_N), \tag{4}$$

where H_N is a constant. For $H_N(x_k)''$ and $H_N(x_k)'$ we get the expressions

$$H_N(x_k)' = H_N \Pi_{i \neq k}(x_k - x_i), H_N(x_k)'' = 2H_N \Pi_{i \neq k}(x_k - x_i) \sum_{i \neq k} \frac{1}{x_k - x_i}. \tag{5}$$

Now we inserting x_k in equation (3):

$$x_k - \frac{H_N(x_k)''}{2H_N(x_k)'} = x_k - \sum_{i \neq k} \frac{1}{x_k - x_i} = 0, \tag{6}$$

i. e. the roots of $H_N(x)$ are stationary points of $V(x_1, \ldots, x_N)$.