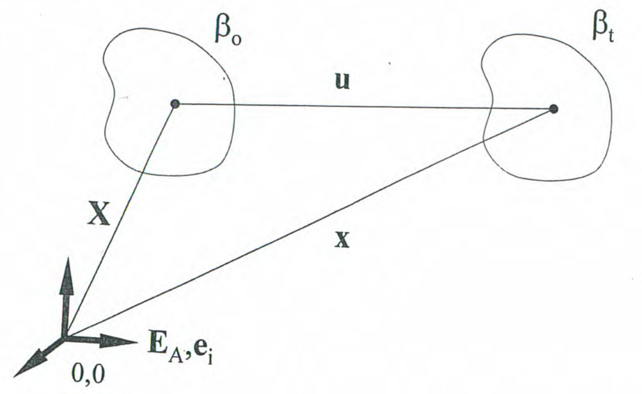


FIGURE 3.1. Schematic illustration of a material body in two configurations—an initial reference configuration at time $t = 0$, denoted as β_o, and a current configuration at time t, denoted as β_t. The position of a material particle, relative to a common origin, is given by $\boldsymbol{X}$ and $\boldsymbol{x}$ in these two configurations, respectively. The displacement $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ and $\boldsymbol{E}_A$ and $\boldsymbol{e}_i$ are orthonormal bases.

According to Truesdell and Noll (1965), there are four basic approaches to describe the kinematics of a continuum: the material, referential, spatial, and relative approaches. In the material approach, motion is described via the particles themselves and time; this approach is not particularly useful in (deformable) solid mechanics. In the referential and spatial approaches, motion is described in terms of time and either the original or current positions of the material particles, respectively. The referential approach, having independent variables $(\boldsymbol{X},t)$ and sometimes called a Lagrangian approach, is particularly useful in elasticity, whereas the spatial approach, having independent variables $(\boldsymbol{x},t)$ and sometimes called an Eulerian approach, is useful in fluid mechanics.[1] Finally, in the relative approach one uses independent variables $(\boldsymbol{x},\tau)$ where τ is a measure of time often related to an intermediate configuration; this approach is useful in viscoelasticity.

Our interest herein is primarily the (pseudo)elastic behavior of soft biosolids, as briefly introduced in section 1.4; thus, we will use the referential approach in most cases. Consequently, let the positions of material particles at time t depend on their original positions, viz.,[2]

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \quad \boldsymbol{x}, \boldsymbol{X} \in V, t \in R. \tag{3.1$_1$}$$

[1] The referential approach was actually introduced by Euler and the spatial approach by D'Alembert.
[2] When we write a function such as $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X},t)$, note that the $\boldsymbol{x}$ on the left-hand side is the value of the function at a particular $\boldsymbol{X}$ and t, whereas the $\boldsymbol{x}$ on the right-hand

Hence, the associated displacement field is given by

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X}. \tag{3.1$_2$}$$

Because we will be interested primarily in the motion of individual material particles, it is useful to consider what happens to generic differential line segments (that connect two such particles) as a body passes from one configuration to another. Hence, let $d\boldsymbol{x}$ be an oriented differential line segment in β_t that was originally $d\boldsymbol{X}$ (having a different magnitude and direction in general) in β_o. A fundamental question then is, How do we relate these two differential position vectors? Recall from Chapter 2 that a second-order tensor transforms a vector into a new vector. Hence, in direct and Cartesian component notations, at each time t, let

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X}, \quad dx_i = F_{iA} dX_A \tag{3.2}$$

where $\boldsymbol{F}$ is a second-order tensor that accomplishes the desired transformation; it is called the deformation gradient.[3] *$\boldsymbol{F}$ will prove to be a fundamental measure of the deformation, from which will come measures of area and volume changes, strain, strain-rate, interrelations between different measures of stress, etc.* Because $\boldsymbol{x}$ is a function of $\boldsymbol{X}$, at each fixed time t, the chain rule requires

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \cdot d\boldsymbol{X}, \quad dx_i = \frac{\partial x_i}{\partial X_A} dX_A. \tag{3.3$_1$}$$

Moreover, comparing equations 3.2 and 3.3$_1$ reveals that

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = F_{iA} \boldsymbol{e}_i \otimes \boldsymbol{E}_A, \quad \textit{where} \quad F_{iA} = \frac{\partial x_i}{\partial X_A}, \tag{3.3$_2$}$$

which provides a method for computing the components of $\boldsymbol{F}$ given a referential description of the motion relative to a Cartesian coordinate system. $\boldsymbol{F}$ is called a two-point tensor because its dyad consists of bases from two coordinate systems;[4] contrast this to the so-called one-point tensors in Chapter 2 (e.g., equation 2.48). All tensor operations hold for $\boldsymbol{F}$, provided that the bases are delineated: for example, $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ and $\boldsymbol{E}_A \cdot \boldsymbol{E}_B = \delta_{AB}$, but we will not consider $\boldsymbol{e}_i \cdot \boldsymbol{E}_A$.

Assuming equation 3.1$_1$ is invertible, that is, that $\boldsymbol{X}$ can be written as a function of $\boldsymbol{x}$ at a fixed time t, we can alternatively consider

[3] Actually, $\boldsymbol{F}$ is a gradient of the motion, $\boldsymbol{x}(\boldsymbol{X},t)$, but we retain the standard terminology.
[4] Actually, a tensor is a tensor. So-called "two-point tensors" are merely most con-

$$dX = \frac{\partial X}{\partial x} \cdot dx, \quad dX_A = \frac{\partial X_A}{\partial x_i} dx_i, \qquad (3.4)_1$$

with

$$F^{-1} = \frac{\partial X}{\partial x} = F^{-1}_{Ai} E_A \otimes e_i, \quad where \quad F^{-1}_{Ai} = \frac{\partial X_A}{\partial x_i}. \qquad (3.4)_2$$

It is important to observe that position vectors dx can be mapped from dX via a rigid body motion (i.e., a translation and/or rotation), a "deformation" (i.e., extension and shear), or a combination of both. Indeed, it can be shown that F can be decomposed via

$$F = R \cdot U = V \cdot R \qquad (3.5)$$

where $R \in Orth^+$ (i.e., $R^{-1} = R^T$ and $det R = 1$) represents the rigid body motion, $U \in Psym$ (i.e., $U^T = U$ and is positive definite) is defined in the reference configuration β_o, and $V \in Psym$ is defined in the current configuration β_t. Referred to Cartesian coordinates,

$$R = R_{iA} e_i \otimes E_A = R_{Ai} E_A \otimes e_i, \quad U = U_{AB} E_A \otimes E_B, \quad V = V_{ij} e_i \otimes e_j. \qquad (3.6)$$

Hence, R is a two-point tensor, whereas U and V are one-point tensors. U and V represent the complete deformation (extension and shear), but are called right and left "stretch" tensors, respectively, because their principal values are the principal stretches (e.g., current divided by reference lengths) experienced by the body at a point. Equation 3.5 can be interpreted, therefore, as "stretch" followed by a "rigid rotation" ($R \cdot U$) or a "rigid rotation" followed by "stretch" ($V \cdot R$); it is called the polar decomposition theorem. See exercise 3.1.

Although F is a fundamental measure of the "deformation," it is not necessarily the best measure for analysis in elasticity. In particular, it is a two-point tensor, it is not symmetric in general, and it may contain rigid body contributions. Two more convenient measures of the deformation are *defined* by (recall equations 2.20 and 2.28)

$$C = F^T \cdot F = U^T \cdot R^T \cdot R \cdot U = U^2 \qquad (3.7)$$

and

$$B = F \cdot F^T = V \cdot R \cdot R^T \cdot V^T = V^2, \qquad (3.8)$$

which are the right and left Cauchy-Green tensors, respectively (an easy way to remember right from left is by noting the position of the fundamental measure F with respect to its transpose). C and B are both one-point, symmetric tensors that are independent of rigid body motion, C being defined in the reference configuration β_o and B in the current configuration β_t. When referred to

$$C = C_{AB} E_A \otimes E_B, \quad where \quad C_{AB} = \frac{\partial x_i}{\partial X_A} \frac{\partial x_i}{\partial X_B}, \qquad (3.9)$$

$$B = B_{ij} e_i \otimes e_j, \quad where \quad B_{ij} = \frac{\partial x_i}{\partial X_A} \frac{\partial x_j}{\partial X_A}. \qquad (3.10)$$

Notice that in the absence of motion (i.e., $x \equiv X$), then $F = I$ and thus $C = I$ and $B = I$. Similarly for a rigid body motion (i.e., $x = Q(t) \cdot X + c(t)$ where Q is a rigid body rotation and c a translation), $F = Q$, but $C = I$ and $B = I$ again (because $Q \in Orth$). It is often convenient, therefore, to define additional measures of deformation, called *strain* tensors, that equal 0 when there is no deformation. Two of the most commonly used strain tensors are defined as

$$E = \frac{1}{2}(C - I), \quad e = \frac{1}{2}(I - B^{-1}), \qquad (3.11)$$

and are called the Green (or St. Venant or Lagrangian) and Almansi (or Hamel or Eulerian) strain tensors, respectively; both are one-point, symmetric, and independent of rigid body motion. Furthermore, because $x = X + u$, equations 3.3 and 3.4 yield

$$F = I + H, \quad where \quad H = \frac{\partial u}{\partial X} \qquad (3.12)$$

and

$$F^{-1} = I - h, \quad where \quad h = \frac{\partial u}{\partial x}. \qquad (3.13)$$

H and h are *displacement* gradient tensors referred to β_o and β_t, respectively. Combining equations 3.11 with 3.12 and 3.13 yields alternate representations of the Green and Almansi strain tensors, that is

$$E = \frac{1}{2}(H + H^T + H^T \cdot H) \qquad (3.14)$$

and

$$e = \frac{1}{2}(h + h^T - h^T \cdot h). \qquad (3.15)$$

These representations are not particularly useful in analytical formulations (equations 3.11 being preferred), but they are useful in some finite element analyses. Finally, note that if the deformation and the rigid body rotations are both small (i.e., $F \simeq I$ and $R \simeq I$), then $\partial u/\partial X \simeq \partial u/\partial x$ and the quadratic terms in equations 3.14 and 3.15 become negligible in comparison to the linear terms. This leads to the definition of the so-called infinitesimal

$$\varepsilon = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) \simeq \frac{1}{2}(\boldsymbol{h} + \boldsymbol{h}^T), \tag{3.16}$$

whereby it is seen that $\boldsymbol{\varepsilon}$ is only an approximate measure of particular measures of strain, unlike $\boldsymbol{E}$ and $\boldsymbol{e}$ which are exact within the context of the continuum theory. Noting that $\boldsymbol{\varepsilon}$ can also be written as (from equation 3.12)

$$\varepsilon = \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}^T) - \boldsymbol{I}, \tag{3.17}$$

reveals important characteristics that are not always appreciated. First, because $\boldsymbol{\varepsilon}$ depends linearly on $\boldsymbol{F}$, it contains rigid body information (which is why $\boldsymbol{\varepsilon}$ is a useful measure of strain only if the rigid body rotations are truly small). Second, $\boldsymbol{\varepsilon}$ is actually a two-point tensor, albeit approximately one point if $\boldsymbol{x} \simeq \boldsymbol{X}$.

Next, let us reconsider the mapping of a differential position vector $d\boldsymbol{X}$ into $d\boldsymbol{x}$ by writing each in terms of their magnitudes and directions. That is, let $d\boldsymbol{X} = dS\boldsymbol{M}$ and $d\boldsymbol{x} = ds\boldsymbol{m}$ where $\boldsymbol{M}$ and $\boldsymbol{m}$ are unit vectors in β_o and β_t, and $dS = |d\boldsymbol{X}|$ and $ds = |d\boldsymbol{x}|$. From equation 3.2 we see that $\Lambda\boldsymbol{m} = \boldsymbol{F}\cdot\boldsymbol{M}$ where $\Lambda = ds/dS$ is called a stretch ratio (i.e., current divided by original length). From this equation one can calculate the orientation $\boldsymbol{m}$ that any differential line segment will have in β_t, given its orientation $\boldsymbol{M}$ in β_o, the deformation gradient $\boldsymbol{F}$, and its stretch Λ. To find Λ independently, note that

$$d\boldsymbol{x}\cdot d\boldsymbol{x} = ds^2(\boldsymbol{m}\cdot\boldsymbol{m}) = ds^2 \tag{3.18$_1$}$$

and, by using equation 3.2 again, that

$$d\boldsymbol{x}\cdot d\boldsymbol{x} = (\boldsymbol{F}\cdot d\boldsymbol{X})^T\cdot(\boldsymbol{F}\cdot d\boldsymbol{X}) = dS^2\boldsymbol{M}\cdot(\boldsymbol{F}^T\cdot\boldsymbol{F})\cdot\boldsymbol{M}, \tag{3.18$_2$}$$

wherein we used the "trick" that, in direct notation, a vector equals its transpose: that is, $d\boldsymbol{x} \equiv d\boldsymbol{x}^T = (\boldsymbol{F}\cdot d\boldsymbol{X})^T = d\boldsymbol{X}\cdot\boldsymbol{F}^T$ by equation 2.20. Combining equations 3.18 yields

$$\Lambda^2 = \left(\frac{ds}{dS}\right)^2 = \boldsymbol{M}\cdot\boldsymbol{C}\cdot\boldsymbol{M} \tag{3.19}$$

which allows Λ to be found given $\boldsymbol{F}$ (because $\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F}$) and the original orientation $\boldsymbol{M}$ of the line segment of interest. In Chapter 5, we will see that equation 3.19 is very useful in the analysis of experimental data.

If we consider the mapping of two differential line segments (Figure 3.2), we can also calculate the angle (change) between them. That is, let $d\boldsymbol{X}^{(1)}$ and $d\boldsymbol{X}^{(2)}$ at a point be mapped into $d\boldsymbol{x}^{(1)}$ and $d\boldsymbol{x}^{(2)}$ by the same $\boldsymbol{F}$; here superscripts (1) and (2) denote vectors one and two, not contravariant components or exponents. Assuming that the original angle Θ is known, which is to say the orientations $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$, the current angle θ can be calcu-

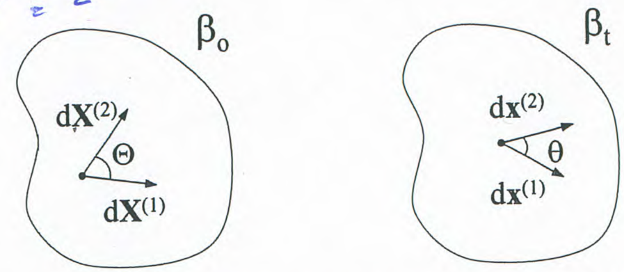


FIGURE 3.2. Considering the displacement of a single material particle is not sufficient for quantifying motion. For example, all points in a rigid body displace (the same) and yet there is no strain; in contrast, points along a fixed edge do not displace even in a deformable body under strain. Of prime importance, therefore, is relative motion between material particles, convenient pairings of which are differential line segments. This schema shows the mapping of two oriented differential line segments from a reference to a current configuration. Each vector can change its length and/or orientation via $d\boldsymbol{x} = \boldsymbol{F}\cdot d\boldsymbol{X}$.

$$d\boldsymbol{x}^{(1)}\cdot d\boldsymbol{x}^{(2)} = ds^{(1)}ds^{(2)}\cos\theta \tag{3.20$_1$}$$

or

$$\cos\theta = \frac{\boldsymbol{M}^{(1)}\cdot\boldsymbol{C}\cdot\boldsymbol{M}^{(2)}}{(ds/dS)^{(1)}(ds/dS)^{(2)}} = \frac{\boldsymbol{M}^{(1)}\cdot\boldsymbol{C}\cdot\boldsymbol{M}^{(2)}}{(\boldsymbol{M}^{(1)}\cdot\boldsymbol{C}\cdot\boldsymbol{M}^{(1)})^{1/2}(\boldsymbol{M}^{(2)}\cdot\boldsymbol{C}\cdot\boldsymbol{M}^{(2)})^{1/2}}. \tag{3.20$_2$}$$

A similar result in terms of the Green strain $\boldsymbol{E}$ is obtained easily since $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$. For example, for the special case where $\boldsymbol{M}^{(1)} = \boldsymbol{E}_1$ and $\boldsymbol{M}^{(2)} = \boldsymbol{E}_2$, we have (recalling equation 2.49)

$$\cos\theta = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}. \tag{3.20$_3$}$$

Thus, this angle change in finite strain depends on both shear strain E_{12} and extensional strain E_{11} and E_{22}. This result is fundamentally different from that for the linearized strain tensor $\boldsymbol{\varepsilon}$ wherein shear components are defined as angle changes (which is recovered by equation 3.20_3 in terms of $\boldsymbol{E}$ if one notes that for $\boldsymbol{F} \simeq \boldsymbol{I}$ the extensional components are small in comparison to 1). Coupling of the effects of extension and shear is common in finite elasticity as will be seen subsequently in specific problems.

Just as it is useful to understand how differential position vectors are mapped from reference to current configurations, it is also useful to quantify how differential areas and volumes are mapped. Let us first con-

Recall that the scalar triple product (exercise 2.6) allows one to calculate a volume from three appropriate vectors. Let us consider, therefore, three differential position vectors $d\boldsymbol{X}^{(1)}$, $d\boldsymbol{X}^{(2)}$ and $d\boldsymbol{X}^{(3)}$, each of which are mapped by $\boldsymbol{F}$ into corresponding vectors $d\boldsymbol{x}^{(1)}$, $d\boldsymbol{x}^{(2)}$, and $d\boldsymbol{x}^{(3)}$. Hence, using the fundamental definition of the determinant (see Bowen, 1989, p. 230), we have

$$\begin{aligned} dv &= d\boldsymbol{x}^{(1)} \cdot (d\boldsymbol{x}^{(2)} \times d\boldsymbol{x}^{(3)}) \\ &= \boldsymbol{F} \cdot d\boldsymbol{X}^{(1)} \cdot (\boldsymbol{F} \cdot d\boldsymbol{X}^{(2)} \times \boldsymbol{F} \cdot d\boldsymbol{X}^{(3)}) \\ &= (\det \boldsymbol{F}) d\boldsymbol{X}^{(1)} \cdot (d\boldsymbol{X}^{(2)} \times d\boldsymbol{X}^{(3)}) \\ &= (\det \boldsymbol{F}) dV \end{aligned} \tag{3.21}$$

which reveals that the det$\boldsymbol{F}$ (= dv/dV) maps original differential volumes into current ones, again reinforcing that $\boldsymbol{F}$ is a fundamental measure of the (finite) deformation. Note, too, that det$\boldsymbol{F}$ is often denoted by J, and that $J > 0$ herein. Moreover, if the deformation is *isochoric* (i.e., volume preserving), det$\boldsymbol{F} = 1$ and the material is said to have behaved incompressibly.

Likewise, consider the mapping of differential areas dA from a reference configuration β_o to areas da in a current configuration β_t (Figure 3.3). Although we could use the cross product to define and relate these areas (e.g., $d\boldsymbol{X}^{(2)} \times d\boldsymbol{X}^{(3)} = dA\boldsymbol{N}$, where $\boldsymbol{N}$ is an outward unit normal vector), we can alternatively exploit equation 3.21. Hence, using equation 3.2,

$$dv = d\boldsymbol{x}^{(1)} \cdot (\boldsymbol{n}da) = d\boldsymbol{X}^{(1)} \cdot \boldsymbol{F}^T \cdot (\boldsymbol{n}da) \tag{3.22}$$

whereas from equation 3.21 (with $J \equiv \det\boldsymbol{F}$),

$$dv = JdV = J(d\boldsymbol{X}^{(1)} \cdot (\boldsymbol{N}dA)). \tag{3.23}$$

Hence, equating equations 3.22 and 3.23, we find that $\boldsymbol{F}^T \cdot \boldsymbol{n}da = J\,\boldsymbol{N}dA$, or, as it is commonly written,

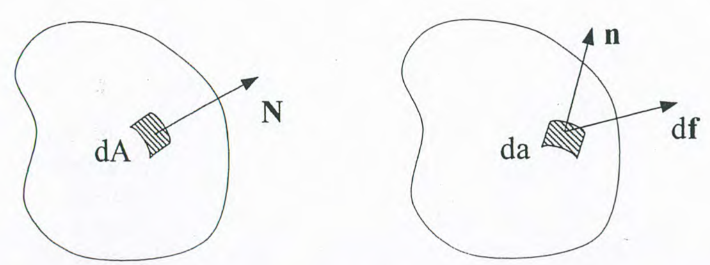


FIGURE 3.3. Similar to Figure 3.2, except for the mapping of a differential area dA into da, which have outward unit normal vectors $\boldsymbol{N}$ and $\boldsymbol{n}$, respectively. Also shown is a differential force $d\boldsymbol{f}$ that acts over da, from which we can define a traction vector $\boldsymbol{T}^{(n)} = d\boldsymbol{f}/da$ (note: the traction vector is sometimes referred to as a stress vector and the definition of $\boldsymbol{T}^{(n)}$ requires taking the limit of $\Delta \boldsymbol{f}/\Delta a$).

5

$$\boldsymbol{n}da = J\boldsymbol{N}dA \cdot \boldsymbol{F}^{-1} \leftrightarrow \boldsymbol{N}dA = \frac{1}{J}\boldsymbol{n}da \cdot \boldsymbol{F}, \tag{3.2}$$

which are known as Nanson's relations. Again we see that $\boldsymbol{F}$, and its det minant, is fundamental to describing the deformation. Nanson's relatio are particularly important in the definition and calculation of stresses, whi are measures of forces acting over oriented areas.

Next, it will prove useful to consider velocity, acceleration, velocity g dients, strain-rates, etc., that is, measures of time-dependent motions ex rienced by material particles within the body of interest. Indeed, one of fundamental laws of mechanics, Newton's second law, relates acceleratio to the forces that cause them in special (i.e., inertial) frames of referen Quantifying accelerations and associated measures is thereby fundamen to mechanics.

Simply put, velocity $\boldsymbol{v}$ is the time rate-of-change of position, and acc eration $\boldsymbol{a}$ is the time rate-of-change of velocity; both are vectors. Rec too, that we can use either a referential or a spatial description of moti The referential description is the most intuitive, and typically the one u in dynamics and solid mechanics. In this approach, we let the current p tion $\boldsymbol{x}$ of a material particle depend on the reference position $\boldsymbol{X}$ and ti t. Consequently,

$$\boldsymbol{v}(t) = \frac{d}{dt}(\boldsymbol{x}(\boldsymbol{X},t)) = \frac{d}{dt}(\boldsymbol{u}(\boldsymbol{X},t)) \quad (3$$

and

$$\boldsymbol{a}(t) = \frac{d^2}{dt^2}(\boldsymbol{x}(\boldsymbol{X},t)) = \frac{d^2}{dt^2}(\boldsymbol{u}(\boldsymbol{X},t)), \quad (3$$

where $\boldsymbol{u} = \boldsymbol{x}(\boldsymbol{X},t) - \boldsymbol{X}$; being the reference position, $\boldsymbol{X}$ does not chang time. Thus, referential $\boldsymbol{v}$ and $\boldsymbol{a}$, for a given particle, depend only on t and their original position. In contrast, in a spatial approach, which independent variables $\boldsymbol{x}$ and t and is typically used in fluid mechanics have

$$\boldsymbol{v}(\boldsymbol{x},t) = \frac{d\boldsymbol{x}}{dt}, \quad \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt}, \quad (3$$

wherein we see that in the spatial approach the acceleration has two tributions, one local ($\partial\boldsymbol{v}/\partial t$) and one convective. The latter can be wri as $\boldsymbol{L} \cdot \boldsymbol{v}$, where $\boldsymbol{L}$ (= $\partial\boldsymbol{v}/\partial\boldsymbol{x}$) is called the *velocity gradient* tensor. Reca that a second-order tensor transforms one vector into another, $\boldsymbol{L}$ transf the differential position vector $d\boldsymbol{x}$ into the velocity vector $d\boldsymbol{v}$ at each t (i.e., $d\boldsymbol{v} = \boldsymbol{L} \cdot d\boldsymbol{x}$, which shows that a tensor can change not only the

Relation (2.83) describes a strain measure in the direction of $\mathbf{a}$ at place $\mathbf{x} \in \Omega$. We have introduced the commonly used *symmetric* strain tensor $\mathbf{e}$, which is well-known as the **Euler-Almansi strain tensor**.

Since the strain tensors $\mathbf{b}$, $\mathbf{e}$ and their inverse operate on the spatial vectors $\mathbf{a}, \mathbf{x}$, we call them **spatial strain tensors**.

We now consider the case $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{I}$. From the relation $(2.58)_2$ we know that the distance between any two neighboring points $\mathbf{X}$ and $\mathbf{Y}$ located in the reference configuration is $d\varepsilon = |\mathbf{Y} - \mathbf{X}|$. On the other hand the distance between the associated neighboring points $\mathbf{x}$ and $\mathbf{y}$ located in the current configuration is given via $(2.63)_3$, i.e. $\lambda d\varepsilon = |\mathbf{y} - \mathbf{x}|$. For $\mathbf{C} = \mathbf{I}$ we conclude from $(2.64)_4$ that the line element is unstretched, i.e. $\lambda = 1$, and consequently $d\varepsilon = |\mathbf{y} - \mathbf{x}| = |\mathbf{Y} - \mathbf{X}|$. Hence, the distance between any two points is unchanged during such a motion. This means that there is *no* relative motion of points under χ. Since $\mathbf{C} = \mathbf{I}$ we conclude additionally from relation (2.69) that the strain tensor $\mathbf{E}$ vanishes identically, which means that the body does not change its size and shape (no changes in distances and angles).

This particular motion, which preserves the distance between any pair of points of a continuum body, is called a **rigid-body motion** and is dealt with in more detail in Section 5.2. Hence, a rigid-body motion induces no strains and consequently no stresses. A body which is only able to undergo a rigid-body motion is said to be a **rigid body**. The idealization that a body is rigid is often considered in engineering dynamics.

Push-forward, pull-back operation. As already seen, vector and tensor-valued quantities may be resolved along triads of basis vectors belonging to either the reference or the current configuration. Additionally, there are two-point tensors which are associated with both configurations, one example being the deformation gradient (2.39). The transformations between material and spatial quantities are typically called a **push-forward operation** and a **pull-back operation** (familiar in differential geometry) and are denoted by short-hand $\chi_*(\bullet)$ and $\chi_*^{-1}(\bullet)$, respectively. In the literature the pull-back operation is often written as $\chi^*(\bullet)$.

In particular, a *push-forward* is an operation which transforms a vector or tensor-valued quantity based on the reference configuration to the current configuration. Since the Euler-Almansi strain tensor $\mathbf{e}$ is defined with respect to spatial coordinates we can compute it as a push-forward of the Green-Lagrange strain tensor $\mathbf{E}$, which is given in terms of material coordinates. From eq. (2.83) we conclude, using definition (2.69), that

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}) = \mathbf{F}^{-\mathrm{T}}[\frac{1}{2}\mathbf{F}^{\mathrm{T}}(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1})\mathbf{F}]\mathbf{F}^{-1}$$

A *pull-back* is an inverse operation, which transforms a vector or tensor-valued quantity based on the current configuration to the reference configuration. Similarly to the above, the pull-back of $\mathbf{e}$ is

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}) = \mathbf{F}^{\mathrm{T}}[\frac{1}{2}\mathbf{F}^{-\mathrm{T}}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})\mathbf{F}^{-1}]\mathbf{F} \\ &= \mathbf{F}^{\mathrm{T}}[\frac{1}{2}(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1})]\mathbf{F} = \mathbf{F}^{\mathrm{T}}\mathbf{e}\mathbf{F} \\ &= \chi_*^{-1}(\mathbf{e}) \ . \end{aligned} \tag{2.85}$$

As can be seen from eqs. (2.84) and (2.85) the transformations are based on multiplications by one description of the deformation gradient, i.e. $\mathbf{F}, \mathbf{F}^{-1}, \mathbf{F}^{\mathrm{T}}, \mathbf{F}^{-\mathrm{T}}$. Which form of the deformation gradient we have to take depends on the tensor to be transformed.

Following MARSDEN and HUGHES [1994] we indicate *covariant tensors* by $(\bullet)^\flat$ and *contravariant tensors* by $(\bullet)^\sharp$ (for the notions 'covariant' and 'contravariant' the reader is referred to Section 1.6). The push-forward and pull-back operations on *covariant* second-order tensors (such as $\mathbf{E}^\flat$, $\mathbf{C}^\flat$, $\mathbf{e}^\flat$, $(\mathbf{b}^{-1})^\flat$) are according to

$$\chi_*(\bullet)^\flat = \mathbf{F}^{-\mathrm{T}}(\bullet)^\flat\mathbf{F}^{-1} \ , \qquad \chi_*^{-1}(\bullet)^\flat = \mathbf{F}^{\mathrm{T}}(\bullet)^\flat\mathbf{F} \ . \tag{2.86}$$

An example was given previously in eqs. (2.84) and (2.85), i.e. $\mathbf{e} = \chi_*(\mathbf{E}^\flat)$ and $\mathbf{E} = \chi_*^{-1}(\mathbf{e}^\flat)$, which provide the relationships between the material and spatial quantities, i.e. $\mathbf{e} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1}$ and $\mathbf{E} = \mathbf{F}^{\mathrm{T}}\mathbf{e}\mathbf{F}$, respectively.

However, the push-forward and pull-back operations on *contravariant* second-order tensors (such as $(\mathbf{C}^{-1})^\sharp$, $\mathbf{b}^\sharp$ and most of the common stress tensors) are according to

$$\chi_*(\bullet)^\sharp = \mathbf{F}(\bullet)^\sharp\mathbf{F}^{\mathrm{T}} \ , \qquad \chi_*^{-1}(\bullet)^\sharp = \mathbf{F}^{-1}(\bullet)^\sharp\mathbf{F}^{-\mathrm{T}} \ . \tag{2.87}$$

In the following chapters we use *covariant strain tensors* in combination with *contravariant stress tensors*.

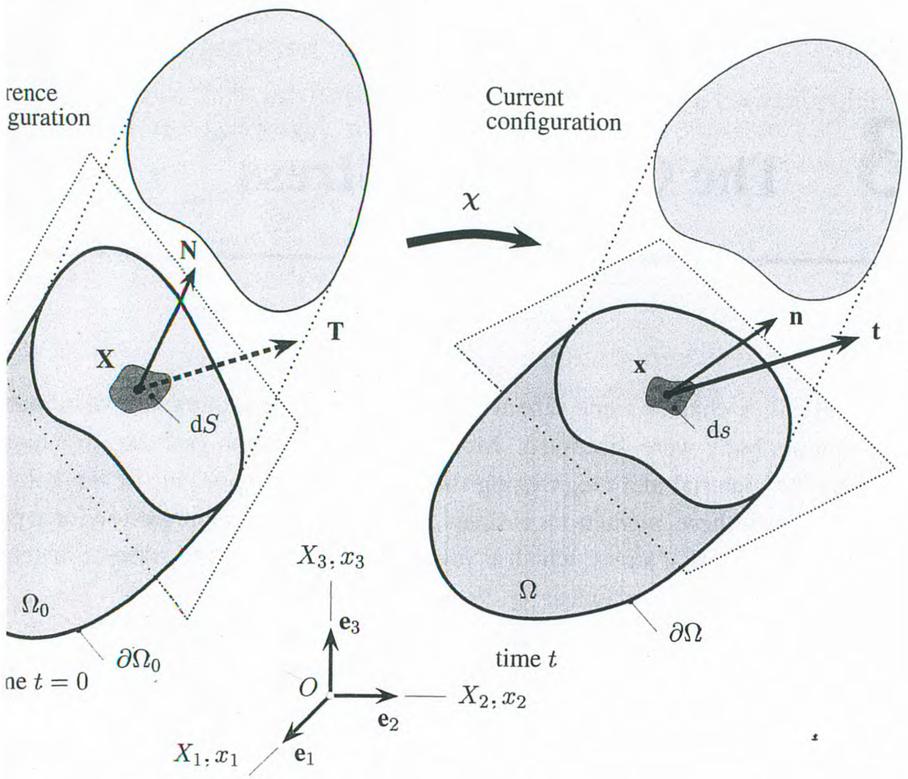
For completeness we write down the push-forward and pull-back operations on *covariant vectors*, i.e.

$$\chi_*(\bullet)^\flat = \mathbf{F}^{-\mathrm{T}}(\bullet)^\flat \ , \qquad \chi_*^{-1}(\bullet)^\flat = \mathbf{F}^{\mathrm{T}}(\bullet)^\flat \ , \tag{2.88}$$

and of *contravariant vectors*, i.e.

$$\chi_*(\bullet)^\sharp = \mathbf{F}(\bullet)^\sharp \ , \qquad \chi_*^{-1}(\bullet)^\sharp = \mathbf{F}^{-1}(\bullet)^\sharp \ . \tag{2.89}$$

Finally we provide the so-called **Piola transformation** of a spatial vector field



3.1 Traction vectors acting on infinitesimal surface elements with outward unit normals.

e postulate that arbitrary forces act on parts or the whole of the boundary surface d *external forces*), and on an (imaginary) surface within the interior of that body d *internal forces*) in some distributed manner.

t the body now be cut by a plane surface which passes any given point $\mathbf{x} \in \Omega$ patial coordinates x_a at time t. As illustrated in Figure 3.1, the plane surface sep- the deformable body into two portions. We focus attention on that part of the *free* nuum) body lying on the tail of a unit vector $\mathbf{n}$ at $\mathbf{x}$, directed along the outward l to an infinitesimal spatial surface element $\mathrm{d}s \in \partial\Omega$. Since we consider inter- of the two portions, forces are transmitted across the (internal) plane surface. note an infinitesimal **resultant (actual) force** acting on a surface element as $\mathrm{d}\mathbf{f}$. ırposes which will be made clear in Section 4.3 we omit distributed so-called **ant couples** not occurring in the classical formulation of continuum mechanics.

physical space with boundary surface $\partial\Omega_0$. The quantities $\mathbf{x}$ (with spatial coordinates x_a), $\mathrm{d}s$ and $\mathbf{n}$ which are associated with the current configuration of the body are denoted by $\mathbf{X}$ (with material coordinates X_A), $\mathrm{d}S$ and $\mathbf{N}$ when they are referred to the reference configuration.

According to Figure 3.1 we claim that for every surface element

$$\mathrm{d}\mathbf{f} = \mathbf{t}\mathrm{d}s = \mathbf{T}\mathrm{d}S \ , \tag{3.1}$$

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}) \ , \qquad \mathbf{T} = \mathbf{T}(\mathbf{X}, t, \mathbf{N}) \ . \tag{3.2}$$

Here, $\mathbf{t}$ represents the **Cauchy** (or **true**) **traction vector** (force measured per unit surface area defined in the *current* configuration), exerted on $\mathrm{d}s$ with outward normal $\mathbf{n}$. The vector $\mathbf{T}$ represents the **first Piola-Kirchhoff** (or **nominal**) **traction vector** (force measured per unit surface area defined in the *reference* configuration), and points in the *same* direction as the Cauchy traction vector $\mathbf{t}$. The (pseudo) traction vector $\mathbf{T}$ does not describe the actual intensity. It acts on the region Ω and is, in contrast to the Cauchy traction vector $\mathbf{t}$, a function of the referential position $\mathbf{X}$ and the outward normal $\mathbf{N}$ to the boundary surface $\partial\Omega_0$. This circumstance is indicated in Figure 3.1 in the form of a dashed line for $\mathbf{T}$. Relationship (3.2) is **Cauchy's postulate**.

The vectors $\mathbf{t}$ and $\mathbf{T}$ that act across the surface elements $\mathrm{d}s$ and $\mathrm{d}S$ with respective normals $\mathbf{n}$ and $\mathbf{N}$ are referred to as **surface tractions** or in some texts as **contact forces**, **stress vectors** or just **loads**. Typical surface tractions are contact and friction forces or are caused by liquids or gases, for example, water or wind.

Cauchy's stress theorem. There exist unique second-order tensor fields $\boldsymbol{\sigma}$ and $\mathbf{P}$ so that

$$\left.\begin{aligned} \mathbf{t}(\mathbf{x}, t, \mathbf{n}) &= \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} \quad \text{or} \quad t_a = \sigma_{ab} n_b \ ; \\ \mathbf{T}(\mathbf{X}, t, \mathbf{N}) &= \mathbf{P}(\mathbf{X}, t)\mathbf{N} \quad \text{or} \quad T_a = P_{aA} N_A \end{aligned}\right\} \tag{3.3}$$

(the proof is omitted), where $\boldsymbol{\sigma}$ denotes a *symmetric* spatial tensor field called the **Cauchy** (or **true**) **stress tensor** (or simply the **Cauchy stress**), while $\mathbf{P}$ characterizes a tensor field called the **first Piola-Kirchhoff** (or **nominal**) **stress tensor** (or simply the **Piola stress**). The index notation in relation (3.3) reveals that $\mathbf{P}$, like $\mathbf{F}$, is a two-point tensor in which one index describes *spatial coordinates* x_a, and the other *material coordinates* X_A. In Section 4.3, p. 147, we establish that the Cauchy stress tensor $\boldsymbol{\sigma}$ is symmetric, under the assumption that resultant couples are neglected.

Relation (3.3), which combines the surface traction with the stress tensor, is one of the most important axioms in continuum mechanics and is known as **Cauchy's stress**

An immediate consequence of (3.3) is the following relationship between $\mathbf{t}$, $\mathbf{T}$ and the corresponding normal vectors, i.e.

$$\mathbf{t}(\mathbf{x},t,\mathbf{n}) = -\mathbf{t}(\mathbf{x},t,-\mathbf{n}) \qquad \text{or} \qquad \mathbf{T}(\mathbf{X},t,\mathbf{N}) = -\mathbf{T}(\mathbf{X},t,-\mathbf{N}) \tag{3.4}$$

for all unit vectors $\mathbf{n}$ and $\mathbf{N}$. This is known as **Newton's (third) law of action and reaction** (see Figure 3.2).

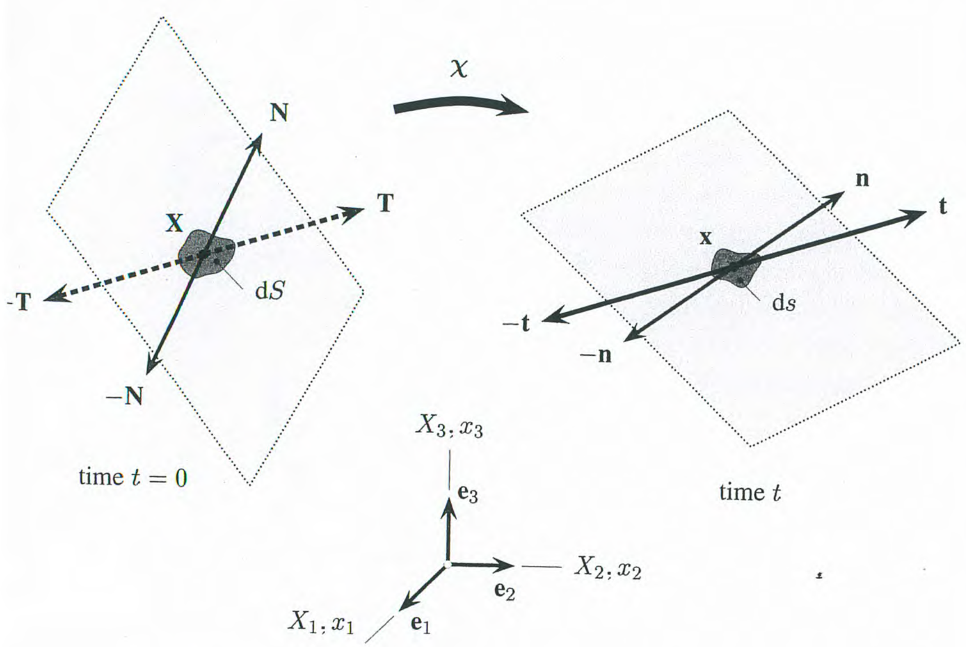


Figure 3.2 Newton's (third) law of action and reaction.

To write Cauchy's stress theorem, for example $(3.3)_1$, in the more convenient matrix tation which is useful for computational purposes, we have

$$[\mathbf{t}] = [\boldsymbol{\sigma}][\mathbf{n}] \quad , \tag{3.5}$$

$$[\mathbf{t}] = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad , \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad , \qquad [\mathbf{n}] = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad , \tag{3.6}$$

ere $[\boldsymbol{\sigma}]$ is usually called the **(Cauchy) stress matrix**.

Finally, we find the relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the first Piola-chhoff stress tensor $\mathbf{P}$. From eq. (3.1) we obtain with eqs. (3.3) and (3.2) the impor-t transformation

$$\boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}ds = \mathbf{P}(\mathbf{X},t)\mathbf{N}dS \quad . \tag{3.7}$$

Using Nanson's formula, i.e. eq. (2.55), $\mathbf{P}$ may be written in the form

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} \qquad \text{or} \qquad P_{aA} = J\sigma_{ab}F_{Ab}^{-1} \quad . \tag{3.8}$$

The passage from $\boldsymbol{\sigma}$ to $\mathbf{P}$ and back is known as the *Piola transformation* (compare with the definition on p. 83). Strictly speaking, in order to obtain the two-point tensor $\mathbf{P}$, with components P_{aA}, we have performed a Piola transformation on the second index of tensor $\boldsymbol{\sigma}$, with components σ_{ab}.

For convenience, we omit subsequently the arguments of the tensor quantities. The explicit expression for the symmetric Cauchy stress tensor results as the inverse of relation (3.8), i.e.

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}} = \boldsymbol{\sigma}^{\mathrm{T}} \qquad \text{or} \qquad \sigma_{ab} = J^{-1}P_{aA}F_{bA} = \sigma_{ba} \quad , \tag{3.9}$$

which necessarily implies

$$\mathbf{P}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{P}^{\mathrm{T}} \quad . \tag{3.10}$$

Consequently, the second-order tensor $\mathbf{P}$ is, in general, *not* symmetric and has nine independent components P_{aA}.

EXAMPLE 3.1 A deformation of a body is described by

$$x_1 = -6X_2 \quad , \qquad x_2 = \frac{1}{2}X_1 \quad , \qquad x_3 = \frac{1}{3}X_3 \quad . \tag{3.11}$$

The Cauchy stress tensor for a certain point of a body is given by its matrix representation as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kN/cm}^2 \quad . \tag{3.12}$$

Determine the Cauchy traction vector $\mathbf{t}$ and the first Piola-Kirchhoff traction vector $\mathbf{T}$ acting on a plane, which is characterized by the outward unit normal $\mathbf{n} = \mathbf{e}_2$ in the *current* configuration.

Solution. From the given deformation (3.11) we find the components of the deformation gradient and its inverse as

$$\begin{bmatrix} 0 & -6 & 0 \end{bmatrix}$$

while $\det\mathbf{F} = J = 1$. The components of the first Piola-Kirchhoff stress tensor read, according to (3.8), as

$$[\mathbf{P}] = J[\boldsymbol{\sigma}][\mathbf{F}^{-\mathrm{T}}] = \begin{bmatrix} 0 & 0 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{kN/cm^2} \quad . \tag{3.14}$$

In order to find the outward unit normal $\mathbf{N}$ in the reference configuration we recall Nanson's formula $\mathbf{N}\mathrm{d}S = J^{-1}\mathbf{F}^{\mathrm{T}}\mathbf{n}\mathrm{d}s$. Hence, with the transpose of matrix $[\mathbf{F}]$, $J = 1$ and knowing that $\mathbf{n} = \mathbf{e}_2$ we find that

$$\mathbf{N}\mathrm{d}S = \frac{1}{2}\mathbf{e}_1\mathrm{d}s \quad , \tag{3.15}$$

thus, $\mathbf{N} = \mathbf{e}_1$. Finally, using Cauchy's stress theorem,

$$[\mathbf{t}] = [\boldsymbol{\sigma}][\mathbf{n}] = \begin{bmatrix} 0 \\ 50 \\ 0 \end{bmatrix} \mathrm{kN/cm^2} \quad , \qquad [\mathbf{T}] = [\mathbf{P}][\mathbf{N}] = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \mathrm{kN/cm^2} \quad , \tag{3.16}$$

i.e. $\mathbf{t} = 50\mathbf{e}_2$ and $\mathbf{T} = 100\mathbf{e}_2$, respectively. As can be seen, $\mathbf{t}$ and $\mathbf{T}$ have the same direction. The magnitude of $\mathbf{T}$ is twice that of $\mathbf{t}$, because, in view of (3.15), the deformed area is half the undeformed area. ■

Stress components. We project the unique Cauchy stress tensor $\boldsymbol{\sigma}$ along an orthonormal set $\{\mathbf{e}_a\}$ of basis vectors; then, according to (1.62), we find that

$$\mathbf{e}_a \cdot \boldsymbol{\sigma}\mathbf{e}_b = \mathbf{e}_a \cdot \mathbf{t}_{\mathbf{e}_b} = \sigma_{ab} \qquad \text{with} \qquad \mathbf{t}_{\mathbf{e}_b} = \boldsymbol{\sigma}\mathbf{e}_b \quad . \tag{3.17}$$

In view of Cauchy's stress theorem $(3.3)_1$, $\mathbf{t}_{\mathbf{e}_b} = \boldsymbol{\sigma}\mathbf{e}_b$, $b = 1, 2, 3$, characterize the three Cauchy traction vectors acting on surface elements whose outward normals point in the directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, respectively. We use the notation $\mathbf{t}_{\mathbf{e}_a} = \mathbf{t}(\mathbf{x}, t, \mathbf{e}_a)$ to indicate explicitly the dependence of the traction vector $\mathbf{t}$ on the basis vectors $\mathbf{e}_a$.

In matrix notation, the columns of $[\boldsymbol{\sigma}]$ can be identified as the components of traction vectors acting on planes perpendicular to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, respectively. We may write

$$\mathbf{t}_{\mathbf{e}_1} = \boldsymbol{\sigma}\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3 \quad , \tag{3.18}$$

$$\mathbf{t}_{\mathbf{e}_2} = \boldsymbol{\sigma}\mathbf{e}_2 = \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3 \quad , \tag{3.19}$$

$$\mathbf{t}_{\mathbf{e}_3} = \boldsymbol{\sigma}\mathbf{e}_3 = \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \tag{3.20}$$

(see the three faces of a cube illustrated in Figure 3.3), characterizing the state of stress at a certain point.

ting that the pressure of the (elastic) fluid is a **mean pressure**.
e three stress fields described for a certain point $\mathbf{x}$ at t (with $\boldsymbol{\sigma}$ constant) corre-
to the states, as illustrated in Figure 3.5.

plane stress state at a certain point is given by the relation

$$\sigma_{13} = \sigma_{23} = \sigma_{33} \equiv 0 \ , \tag{3.59}$$

$_{11}, \sigma_{22}, \sigma_{12}$ are functions of the coordinates x_1 and x_2 only. Consequently, the ection, as represented by $\hat{\mathbf{n}}_3$, is a principal direction of stress with a zero cor- ding principal stress σ_{33}. The other two principal directions acting in a plane l to $\hat{\mathbf{n}}_3$ are inclined at an angle θ with the x_1 and x_2 direction, where

$$\tan 2\theta = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \ . \tag{3.60}$$

orresponding maximum and minimum stresses are given by

$$\frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm [\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2]^{1/2} \ , \tag{3.61}$$

g a biaxial stress state.
e maximum shear stress $\tau_{\max}$ for a plane stress state will be the largest of the values of $(3.48)_2$–$(3.50)_2$ since $\sigma_3 = 0$. The planes of extremal shear stresses ngles of $\pm 45°$ with the planes of the principal stresses.
plane stress state occurs at any unloaded surface in a continuum body and is of al interest.

EXERCISES

For the case of plane stress, show that eq. (3.38) reduces to eq. (3.61).

Assume a plane stress state in a rectangular cube bounded by the planes $x_1 = \pm a, x_2 = \pm b, x_3 = \pm c$. The state of stress at a point with coordinates x_1, x_2, x_3 in the cube is given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \alpha(x_1 - x_2) & \beta x_1^2 x_2 & 0 \\ \beta x_1^2 x_2 & -\alpha(x_1 - x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kN/cm}^2 \ ,$$

where α, β are constants. For a point with coordinates $(a/2, -b/2, 0)$, determine

(a) the principal normal stresses and the associated principal directions,

(b) the planes, characterized by the unit normal $\mathbf{n}$, that give the maximum and minimum shear stresses and the magnitude of the ...

3.4 Alternative Stress Tensors

Numerous definitions and names of stress tensors have been proposed in the literature. Each definition has advantages and disadvantages. In the following we discuss stress tensors used for practical nonlinear analyses. Most of their components do not have a direct physical interpretation.

Often it is convenient to work with the so-called **Kirchhoff stress tensor** $\boldsymbol{\tau}$, which differs from the Cauchy stress tensor by the volume ratio J. It is a contravariant spatial tensor field parameterized by spatial coordinates, and is defined by

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \qquad \text{or} \qquad \tau_{ab} = J\sigma_{ab} \ . \tag{3.62}$$

We introduce further the **second Piola-Kirchhoff stress tensor S** which does not admit a physical interpretation in terms of surface tractions. The contravariant material tensor field is symmetric and parameterized by material coordinates. Therefore, it often represents a very useful stress measure in computational mechanics and in the formulation of constitutive equations, in particular, for solids, as we will see in Chapter 6.

The second Piola-Kirchhoff stress tensor is obtained by the pull-back operation on the contravariant spatial tensor field $\boldsymbol{\tau}^\sharp$ by the motion χ, which is, according to $(2.87)_2$,

$$\mathbf{S} = \chi_*^{-1}(\boldsymbol{\tau}^\sharp) = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}} \qquad \text{or} \qquad S_{AB} = F_{Aa}^{-1}F_{Bb}^{-1}\tau_{ab} \ . \tag{3.63}$$

Hence, the Kirchhoff stress tensor is the push-forward of $\mathbf{S}$, i.e., using $(2.87)_1$,

$$\boldsymbol{\tau} = \chi_*(\mathbf{S}^\sharp) = \mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}} \qquad \text{or} \qquad \tau_{ab} = F_{aA}F_{bB}S_{AB} \ . \tag{3.64}$$

Using eqs. $(3.63)_2$, (3.62) and (3.8) we obtain the Piola transformation relating the two stress fields $\mathbf{S}$ and $\boldsymbol{\sigma}$, i.e.

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \mathbf{F}^{-1}\mathbf{P} = \mathbf{S}^{\mathrm{T}}$$
$$\text{or} \qquad S_{AB} = JF_{Aa}^{-1}F_{Bb}^{-1}\sigma_{ab} = F_{Aa}^{-1}P_{aB} = S_{BA} \ , \tag{3.65}$$

with its inverse,

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}} \qquad \text{or} \qquad \sigma_{ab} = J^{-1}F_{aA}F_{bB}S_{AB} \ . \tag{3.66}$$

From eq. (3.65) we find a fundamental relationship between the first Piola-Kirchhoff stress tensor $\mathbf{P}$ introduced in (3.8) and the symmetric second Piola-Kirchhoff stress ten-