QUANTUM TOPO-DYNAMICS (QTD)

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A generic local quantum spin hamiltonian is introduced that preserves the topological structure of the states it acts upon. Some simple examples are discussed. It is shown that in one dimension the model is capable of producing galilean invariance and confinement. It is further shown in one dimension that the generic QTD hamiltonian can produce relativistic dynamics of massless particles. In any number of dimensions the model is capable also of producing relativistic massive bosons obeying the Klein–Gordon equation as its elementary excitations.

In recent years we have witnessed the growing role of topological excitations in explaining various natural phenomena [1,2].

In this letter we introduce a new set of local quantum spin systems where the topology dominates the dynamics. Namely, the topological structure associated to the physical states is preserved by the dynamics.

After introducing the quantum topo-dynamics (QTD) generic hamiltonian and showing that it can be written in terms of *local* interactions, we produce some simple but interesting results. We show in 1D that these systems can produce galilean invariance and confinement. We show in 1D that an inner motion of an elementary excitation is producible that resembles a massless quantum relativistic particle hitting an impenetrable barrier. We show in any number of dimensions that the models are capable of producing elementary excitations obeying the Klein–Gordon equation and described by a scalar field.

These examples serve to illustrate the richness of the QTD systems. We hope to deal with some of the more intricate aspects of QTD in the very near future.

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Consider a system of spin 1/2 degrees of freedom located on the sites of a d-dimensional lattice. The eigenfunctions of all the σ_i^z span the Hilbert space. Each such function may be described by the set of sites on which the σ_i^z are negative.

We employ the following definitions:

(1) If two Wigner–Seitz cells centered around two lattice sites have at least one common point we say that the corresponding sites are neighbours.

(2) A cluster of positive spins $\{G\}$ is connected either if it consists only of one site or if for any site belonging to $\{G\}$ at least one of its neighbours belongs also to $\{G\}$.

(3) A connected cluster of negative spins is defined in a similar manner, but with nearest neighbours replacing neighbours in definition (2).

The topological properties of a σ_i^z basis function are the properties of the clusters of negative spins describing it.

Our aim is to construct a local hamiltonian, that on one hand changes the clusters but on the other hand preserves the topological properties of the state.

Consider the following d-dimensional generic hamiltonian

$$H = \mathcal{H}\{\sigma_i^z\} + g \sum_i T_\mathrm{p}^i [n(1-\sigma_i^x) + (1-n)\sigma_i^y] T_\mathrm{p}^i , \tag{1}$$

where n is either one or zero and T_p^i is a topology

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preserving projector, that will enable the flipping of a spin, due to the action of σ^x or σ^y, only when this operation does not change the topological properties of the state.

The important observation concerning T_p^i is that it is a local operator depending on spin operators in the near vicinity of the site i. This property may be verified by inspection. The global topological properties of a state are unaltered by the flipping of a given spin if that operation does not change the topological property of the system restricted to the site of the flipped spin and some of its neighbours. (The actual number of neighbour shells depends on the lattice. For example, the relevant shells for the 2D square lattice are the nearest-neighbour shell and the next-nearest-neighbour shell, while for the triangular lattice the nearest-neighbour shell suffices.)

Let us write down some explicit examples for T_p^i. For the one-dimensional lattice it is easily seen that

$$T_p^i = \tfrac{1}{4}[(1+\sigma_{i+1}^z)(1-\sigma_{i-1}^z)+(1-\sigma_{i+1}^z)(1+\sigma_{i-1}^z)] . \tag{2}$$

Consider the two-dimensional triangular lattice. Let $\{k(i)\}$ be the set of nearest neighbours of the site i, enumerated in anticlockwise direction. It may be verified by inspection, that the topological structure is unchanged by flipping the spin i, when two local conditions are met: (a) The number $M(i)$ of neighbours of i where the spin is already negative obeys $1 \leqslant M(i) \leqslant 5$. (b) The neighbouring sites on which the spin is negative form a connected cluster (by themselves).

The following projection operator represents the above conditions.

$$T_p^i = \sum_{k=1}^{6} \sum_{m=k}^{k+4} \prod_{j=k}^{m} \frac{1+\sigma_j}{2} \prod_{l=m\oplus 1}^{k\ominus 1} \frac{1-\sigma_l}{2} , \tag{3}$$

where $\oplus$ and $\ominus$ are addition and subtraction mod 6 (in fact all the integers are defined mod 6).

For the two-dimensional square lattice, let $\{k(i)\}$ be the set of nearest neighbours enumerated in anticlockwise direction and $\{k'(i)\}$ the set of next-nearest neighbours such that $k'(i)$ follows $k(i)$ in anticlockwise direction. The corresponding projection operator is given in this case by

$$T_p^i = \sum_{k=1}^{4} \sum_{m=k}^{k\oplus 2} \prod_{j=k}^{m} \frac{1+\sigma_j}{2} \prod_{j'=k'}^{m'\ominus 1} \frac{1+\sigma_{j'}}{2} \prod_{l=m\oplus 1}^{k\ominus 1} \frac{1-\sigma_l}{2} + \sum_{k'=1}^{4} \prod_{l'\neq k'}^{4} \frac{1+\sigma_{l'}}{2} \prod_{l=1}^{4} \frac{1+\sigma_l}{2} \frac{1-\sigma_{k'}}{2} . \tag{4}$$

We find that T_p^i commutes with σ_i^x and σ_i^y, implying that only one T_p^i is really needed in the definition of H.

(a) 1D galilean continuum limit. Two years ago we considered a special form of the generic hamiltonian (1) [3]

$$H = h \sum_i (1-\sigma_i^2) + g \sum_i \frac{1-\sigma_{i-1}^2}{2}(1+\sigma_i^x)\frac{1+\sigma_{i+1}^2}{2} + \frac{1+\sigma_{i-1}^2}{2}(1+\sigma_i^x)\frac{1-\sigma_{i+1}^2}{2} . \tag{5}$$

Defining $H = hd$ and $g = G/d^2$ where H and G are finite constants and d the lattice distance, we obtained in the continuum limit a Schrödinger equation for the elementary excitations. The elementary excitations are connected regions of negative spin defined by their end points R_1 and R_2 and the equation describing the amplitude $\psi(R_1, R_2)$ for such a region is given by

$$E\psi(R_1,R_2) = -\frac{G}{4}\left(\frac{\partial^2\psi}{\partial R_1^2} + \frac{\partial^2\psi}{\partial R_2^2}\right) + H|R_1 - R_2|\psi . \tag{6}$$

The particle-like solutions of eq. (6) obey galilean invariance. (The boundary condition is $\partial\psi/\partial(R_1 - R_2)|_{R_1=R_2} = 0$.)

(b) 1D massless relativistic particles. Another interesting continuum limit is obtained for the QTD hamiltonian in one dimension by considering $n = 0$ and $\mathcal{H} = 0$. Let $\psi(n_1, n_2)$ be the amplitude of obtaining a region of negative spins with end points n_1, n_2. It is easily shown that

$$H\psi(n_1,n_2) = -\mathrm{i}g\, \mathrm{sign}(n_2 - n_1) \times [\psi(n_1, n_2+1) - \psi(n_1, n_2-1) + \psi(n_1-1, n_2) - \psi(n_1+1, n_2)] \tag{7}$$

with the boundary condition $\psi(n, n) = 0$. Let $g = G/d$, where G is a finite constant and d the lattice constant. In the limit $d \to 0$, eq. (7) becomes

$$H\psi(x_1, x_2) = -\mathrm{i}\,(G/2)\,\mathrm{sign}(x_2 - x_1)$$

$$\times\ [\partial\psi(x_1, x_2)/\partial x_2 - \partial\psi(x_1, x_2)/\partial x_1]\,. \quad (8)$$

Separating variables we obtain in the difference variable $x = x_2 - x_1$

$$E\psi(x) = -\mathrm{i}G\ \mathrm{sign}(x)\partial\psi/\partial x\,. \quad (9)$$

The solution is

$$\psi(x) = \mathrm{sign}(x)\exp[\mathrm{i}\ \mathrm{sign}(x)Px]\,, \quad (10)$$

and the corresponding energy is

$$E(p) = GP\,. \quad (11)$$

The internal motion of the reversed spin region is similar to the motion of a massless quantum relativistic particle scattered off a wall.

(c) d-dimensional massive relativistic bosons. Let us consider now a different limit of the generic QTD hamiltonian that produces massive bosons obeying the Klein–Gordon equation as the elementary equations.

We choose again $n = 0$ and we take the σ^z dependent part $\mathcal{H}\{\sigma^z\}$ to be

$$\mathcal{H}\{\sigma^z\} = -h\sum_i \sigma_i^z + J\sum_{(i,j)\mathrm{n.n.}} \sigma_i^z$$

$$-\ k\sum_{(i,j,k)} \sigma_i^z\sigma_j^z\sigma_k^z\,, \quad (12)$$

where (i,j)n.n is a next nearest pair, (i,j,k) a connected cluster of three sites, the coupling h, J and k are positive and the lattice is assumed to be a d-dimensional cubic lattice.

Let

$$h = 2dA + A' \quad (13)$$

and

$$J = A + J'\,.$$

We consider the case $A \to \infty$ and $k \to \infty$. It is straightforward to show that the finite energy states are in this limit linear combinations of states of $d + 1$ types only. Define $\phi(i)$ as the amplitude for a single-site cluster at the point i and $\psi_j(i, i + \Delta_j)$ as the amplitude of having a nearest-neighbour-pair cluster at the point i and $i + \Delta_j$ along the j axis, where Δ_j is a lattice constant along the j axis.

$$H\phi(i) = [-h + 2dJ]\,\phi(i)$$

$$-\ \mathrm{i}g\sum_j \psi_j(i, i + \Delta_j) + \Delta_j(i, -\Delta_j, i) \quad (14)$$

and

$$H\psi_j(i, i + \Delta_j) = 2[-h + 2dJ]\,\psi_j(i, i + \Delta_j)$$

$$+\ \mathrm{i}g[\phi(i) + \phi(i + \Delta_j)]\,. \quad (15)$$

Define

$$\bar\phi(i, t) = \chi(i)\exp(\mathrm{i}\tfrac{3}{2}\Delta t)\phi(i, t)$$

and

$$\bar\psi_j(i, i + \Delta_j, t) = \chi(i)\exp(\mathrm{i}\tfrac{3}{2}\Delta t)\psi_j(i, i + \Delta_j, t)\,, \quad (16)$$

where t is the time, $\phi(i, t)$ and $\psi_j(i, i + \Delta_j, t)$ are the time dependent amplitudes, $\Delta = -h + 2dJ$ remains finite in the limit $A \to \infty$ and $\chi(i)$ is +1 on the even sublattice and −1 on the odd sublattice. If $g = G/d$ where G remains finite in the limit $d \to 0$, we obtain the time dependent equations

$$\mathrm{i}\frac{\partial}{\partial t}\,\bar\phi(x, t) + \tfrac{1}{2}\Delta\bar\phi(x, t) = -\mathrm{i}G\sum_j \frac{\partial}{\partial x_j}\,\bar\psi_j(x, t) \quad (17)$$

and

$$\mathrm{i}\frac{\partial}{\partial t}\,\bar\psi_j(x, t) - \tfrac{1}{2}\,\Delta\bar\psi_j(x, t) = -\mathrm{i}G\frac{\partial}{\partial x_j}\,\bar\phi(x, t)\,. \quad (18)$$

The above equations lead to the Klein–Gordon equation for the scalar field $\bar\phi$

$$G^2\nabla^2\bar\phi - \partial^2\bar\phi/\partial t^2 + \tfrac{1}{4}\Delta^2\bar\phi = 0\,. \quad (19)$$

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