Aharonov-Bohm Effect for Neutral Particles*

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ABSTRACT

The effective Lagrangian which describes the interaction between a charged particle and a magnetic moment in the non relativistic limit is derived. It is shown that neutral particles with a magnetic moment exhibit the Aharonov-Bohm effect when moving in an external electric field[1]. When applied to a relativistic superconductor the phenomenon is found to be related to the mutual exclusiveness of electric and magnetic condensations.

1 Introduction

Probably the most prominent feature which distinguishes quantum mechanics from classical physics is the existence of topological interference effects. The non-classical nature is manifested through the dependence of the wave function on a phase which accumulates along paths in a force-free region of space. The topological origin is exhibited through the fact that these effects depend only on global properties and are invariant under local deformations of the interfering paths. These two fundamental properties are related and originate from the identification of the field which generates the phase as a gauge field.

The generic topological effect is the abelian Aharonov-Bohm[2] (A-B) effect. The relative phase which controls the interference between two paths is given by the magnetic

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flux enclosed:

$$\exp iS_{AB} = \exp ie \oint \vec{A} \cdot d\vec{x}, \tag{1}$$

where $\vec{A}$ is the vector-potential and e the electric charge of the particle. Clearly the requirement that S_{AB} be invariant under deformations implies vanishing magnetic field and vice versa:

$$\Delta S_{AB} = 0 \Longleftrightarrow \nabla \times \vec{A} = 0, \tag{2}$$

in the region swept by the deformation. The non-relativistic Lagrangian which implies the phase S_{AB} is

$$L = \frac{1}{2}mv^2 + e\vec{A} \cdot \vec{v}, \tag{3}$$

which is clearly the Lagrangian of a charged particle in an external magnetic field. Under a gauge transformation L changes by a total time-derivative:

$$\vec{A} \to \vec{A} + \nabla\Lambda \Longrightarrow\to L + \frac{d}{dt}(e\Lambda). \tag{4}$$

In quantum mechanics the wave function is built by a path integral over $\exp i \int dtL$ so that under the transformation (4):

$$\psi(\vec{x}, t) \to e^{ie\Lambda(\vec{x},t)}\psi(\vec{x}, t). \tag{5}$$

Equation (5) clearly identifies the particle as a charge carrier.

Conversely, neutral particles interact directly with the field-strengths (through non-minimal couplings) so that it would seem that topological phases cannot occur. We shall see below that this conclusion is wrong, since there may be a distinction between a field-free region and a force-free region. More specifically, it will be seen that the laws of electrodynamics and special relativity imply that particles with magnetic form factors interact with charged particles through an effective vector potential and may therefore exhibit in the right circumstances the phenomena characteristic of charged particles in an external gauge-field.

2 A Classical Paradox and Its Resolution

Consider a solenoid located at the point $\vec{R}$ (in the (x, y) plane – all vectors in what follows are two-dimensional) and a charged particle whose coordinate is $\vec{r}$. The vector potential whose source is the solenoid is ($c = \varepsilon_0 = \mu_0 = 1$):

$$\vec{A}(\vec{r} - \vec{R}) = \frac{\vec{\mu}}{2\pi} \times \frac{\vec{r} - \vec{R}}{|\vec{r} - \vec{R}|^2}, \tag{6}$$

where

$$\vec{\mu} = \Phi \hat{n}. \tag{7}$$

$\hat{n}$ is the unit vector perpendicular to the plane and Φ is the magnetic flux. Clearly the magnetic field vanishes outside the solenoid:

$$\nabla \times \vec{A} = 0. \tag{8}$$

Let us endow now the solenoid with a mass M and treat the system as a two-particle system. Since the solenoid is electrically neutral, while the charged particle does not produce a magnetic field when at rest, it would seem that the effective Lagrangian should be:

$$L_{eff} \stackrel{?}{=} \frac{1}{2}mv^2 + \frac{1}{2}MV^2 + e\vec{v} \cdot \vec{A}(\vec{r} - \vec{R}). \tag{9}$$

Equation (9) leads however to a paradox* : when the solenoid moves it generates a non-gauge invariant force on the charged particle:

$$m\dot{\vec{v}} = e\vec{v} \times \vec{B}(\vec{r} - \vec{R}) - e(\vec{V} \cdot \nabla)\vec{A}(\vec{r} - \vec{R}). \tag{10}$$

The first term on the r.h.s. is the correct Lorentz-force and vanishes when $\vec{B} = 0$; the second term is manifestly non-gauge-invariant and non-zero. It is clear that equation (9) cannot be the correct small-velocity limit of the electrodynamic coupling between the charge and the solenoid.

In order to resolve the paradox and identify the missing term in L, we note first that our solenoid may be viewed as a distribution of magnetic moments aligned in the $\hat{n}$-direction such that the linear magnetic moment density if $\vec{\mu}$ (eq. 7). The

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* After ref. (1) was published we were informed by M. Kugler that a related paradox was considered by S. Coleman and J.H. Van Vleck[3]. These authors derived the correct equations of motion to order V^2/c^2 and discussed the conservation of momentum. The effective canonical theory and its quantum mechanical implications were not derived in ref. (3).

field of the solenoid is then the linear superposition of the fields due to the magnetic moments. We further postulate that the forces which bind the magnetic moments and the solenoid are relativistically invariant (a quantum field theory model which realizes the postulate will be analyzed below). Observe now that a magnetic moment is by definition a current distribution (vectors are now 3-dim.!):

$$\vec{j} = \nabla \times \vec{u}; \qquad \int d^3x\vec{u} = \vec{\mu}. \tag{11}$$

When the magnetic moment moves with velocity V Lorentz invariance decrees that a charge density be created, and to order V/c:

$$\rho = \vec{V} \cdot \vec{j}. \tag{12}$$

Thus, the effective Lagrangian of a point-like magnetic moment in an external longitudinal electric field is

$$\begin{aligned} L_{eff} &= \tfrac{1}{2}MV^2 - \textstyle\int \rho A_0 = \tfrac{1}{2}MV^2 - \vec{V} \cdot \textstyle\int A_0 \nabla \times \vec{u} \\ &= \tfrac{1}{2}MV^2 - \vec{V} \cdot \textstyle\int \vec{E} \times \vec{u} \simeq \tfrac{1}{2}MV^2 - \vec{V} \cdot \vec{E} \times \vec{\mu}, \end{aligned} \tag{13}$$

where $E = -\nabla A_0$* is the electric field at the location $\vec{R}$ of the moment. Substituting now for $\vec{E}$ the Coulomb field of a point charge at $\vec{r}$:

$$\vec{E} = \frac{e}{4\pi} \frac{(\vec{R} - \vec{r})}{|\vec{R} - \vec{r}|^3}, \tag{14}$$

we find:

$$L_{eff} = \frac{1}{2}MV^2 - \vec{V} \cdot e\vec{A}(\vec{r} - \vec{R}), \tag{15}$$

where $\vec{A}(\vec{r} - \vec{R})$ is *precisely the vector potential induced by the magnetic moment at the point* $\vec{r}$ according to Ampère's law!

We conclude that the correct charge-magnetic-moment Lagrangian in the non-

* In the gauge $\nabla \cdot \vec{A} = 0$.

relativistic approximation is:

$$L_{eff} = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 + e\vec{A}(\vec{r} - \vec{R}) \cdot (\vec{v} - \vec{V}). \tag{16}$$

Equation (16) resolves the paradox. In fact, it possesses the remarkable property of Galilean invariance! Thus the forces depend only on the relative velocity:

$$m\dot{\vec{v}} = -M\dot{\vec{v}} = e(\vec{v} - \vec{V}) \times \vec{B}(\vec{r} - \vec{R}). \tag{17}$$

Equations (16)-(17) are clearly gauge invariant and in the case of a solenoid produce a vanishing force both on the charge and on the solenoid in any Galilean frame. Gauge invariance is verified easily for gauge functions which depend on $(\vec{r} - \vec{R})$:

$$\vec{A} \rightarrow \vec{A} + \nabla\Lambda(\vec{r} - \vec{R}) \Longrightarrow L \rightarrow L + \frac{d}{dt}(e\Lambda).$$

If the gauge is non translationally invariant the form $(-\vec{V} \cdot \vec{E} \times \vec{\mu})$ should be used. The modifications of the Lagrangian and the equations of motion may be understood better by analyzing the conservation of momentum. The Lagrangian depends only on $(\vec{r} - \vec{R})$ and hence the total canonical momentum is conserved:

$$\vec{p} + \vec{P} = (m\vec{v} + e\vec{A}) + (M\vec{V} - e\vec{A}) = m\vec{v} + m\vec{V} = const. \tag{18}$$

On the other hand:

$$e\vec{A} = \int d^3R\vec{E} \times \vec{u} = \int d^3RA_0\vec{j} = \int d^3RA_0\nabla \times \vec{B} = \int \vec{E} \times \vec{B}. \tag{19}$$

Thus, the total kinetic momentum of the solenoid includes the field momentum:

$$M\vec{V} = \vec{P}_{\rm sol} + \vec{P}_{\rm field}. \tag{20}$$

The latter is "carried along" with the solenoid since the fields $\vec{E}$, $\vec{B}$ are quasi-static and there is no radiation. Equations (19)-(20) are the trace of the retardation effects "inside" the magnetic moment, which previously were taken into account by postulating the correct Lorentz transformation for $\vec{j}$ (eq. 12).

3 The Dirac Equation

We have seen that a classical point-like magnetic moment interacts with an external longitudinal electric field through an effective vector potential $(-\vec{E}\times\vec{\mu})$. We shall now verify this result by deriving it from the Dirac equation.

The Dirac Lagrangian for a neutral magnetic moment is:

$$\mathcal{L} = \bar{\psi}[i\partial\!\!\!/ - M - \frac{1}{2}\mu F_{\mu\nu}\sigma^{\mu\nu}]\psi.$$

In the non relativistic limit, upon using:

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{21}$$

we find (for $F_{0k} = E^k, F_{kl} = 0$):

$$\begin{aligned} i\dot{u} &= Mu + \vec{\sigma}\cdot(\vec{p} - i\mu\vec{E})v \\ -i\dot{v} &= Mv - \vec{\sigma}\cdot(\vec{p} + i\mu\vec{E})u. \end{aligned} \tag{22}$$

Defining now $i\partial_t \simeq M + H_{NR}$ we get:

$$H_{NR} = \vec{\sigma}\cdot(\vec{p} - i\mu\vec{E})\frac{1}{2M}\vec{\sigma}\cdot(\vec{p} + i\mu\vec{E}). \tag{23}$$

Expanding the product of Pauli matrices leads to $(\vec{\mu} \equiv \mu\vec{\sigma})$, $(\nabla\cdot\vec{E} = 0)$:

$$H_{NR} = \frac{1}{2M}(\vec{p} + \vec{E}\times\vec{\mu})^2 - \frac{\mu^2E^2}{2M}. \tag{24}$$

The first term on the r.h.s. is precisely the Hamiltonian derived from the Lagrangian (13). The second term is quadratic in the field and may be neglected for weak fields. It corresponds to an induced electric dipole moment in an external electric field; note that effects due to induced electric moments were disregarded also in the classical treatment.

Thus, apart from the fact that in quantum mechanics $[\mu_i, \mu_k] \neq 0$, the non-relativistic limit of the Dirac theory coincides with the quantized version of the classical theory. The relevance of the non-commutativity of the magnetic-moment components for a spin 1/2 neutral particle will be briefly discussed below. We remark here that a strong polarizing magnetic field will render the spin classical without affecting the orbital motion.

4 The A-B Effect

The Lagrangian (16) possesses a manifest duality property between the charged particle and the magnetic moment. In particular, the effective vector potential of a magnetic moment $\vec{\mu}$ moving in the field of a charged wire will be precisely that of charged particle in the magnetic field of a solenoid. If (for simplicity) $\vec{\mu}$ is polarized in parallel to the charged line the correspondence is made through the substitution:

$$e\Phi \rightarrow -\mu\lambda \tag{25}$$

where Φ is the magnetic flux of the solenoid and λ the linear charge density of the wire.

We thus infer from eqs. (6,7,16,25) that a magnetic moment moving in the field of a charged line experiences no force and exhibits an A-B interference characterized by the phase

$$S_{AB} = -\mu\lambda. \tag{26}$$

In order to get a feeling for the orders of magnitudes involved let us express λ in terms of a characteristic length ℓ:

$$\lambda = e/\ell \tag{27}$$

where e is the proton charge. μ will be given in terms of Bohr magnetons:

$$\mu = \frac{ge}{2M}. \tag{28}$$

Equation (26) then reads (restoring h and c):

$$S_{AB}/2\pi = -\alpha g\hbar/Mc\ell = -g\frac{r_{c\ell}(M)}{\ell} \tag{29}$$

where $r_{c\ell}$ is the classical e.m. radius of a particle of charge e and mass M. $g = 0(1)$

is a gyromagnetic factor* . If the moving magnetic moment is a neutron the relevant mass is the nucleon mass while for an atom the magnetic moment is controlled by the electron mass. Defining ℓ_c as the value needed to get $S_{AB} = -\frac{\pi}{2}$, we find

$$\ell_c(neutron) \simeq 10^{-15}\text{cm} ; \qquad \ell_c(atom) \simeq 10^{-12}\text{cm}. \tag{30}$$

The electric field required to produce a maximal effect on a path of radius = 1cm is thus

$$\begin{aligned} E_{neutron} &\simeq 3 \times 10^8 V/\text{cm} \\ E_{atom} &\simeq 3g \times 10^5 V/\text{cm}. \end{aligned} \tag{31}$$

The phase factor $\exp iS_{AB}$ is clearly a periodic function of S_{AB}. The period is:

$$\Delta(\mu\lambda) = 2\pi. \tag{32}$$

Consider now a superfluid whose constituents are magnetic moments. If the superfluid occupies a non simply connected region of space, and a charge is inserted into the hole the fluid will start to rotate. More precisely, the positively and negatively polarized magnetic moments will rotate in opposite directions. The rotation velocity around a hole of radius r is found by equating the phase to the angular momentum of the rotating particles and we find (for $| S_{AB} | < \pi$):

$$v_{rot} = \frac{S_{AB}}{\pi} \cdot \frac{\hbar}{2Mr}. \tag{33}$$

v is a periodic function of the charge density, with a period given by eq. (32). Finally, let us go back to the Hamiltonian (24) which was derived from the Dirac equation, and estimate the effects due to the non commutativity of the components of $\vec{\mu}$. The A-B phase was derived by assuming that $\vec{\mu}$ was polarized. $\vec{\mu}$ however, satisfies an equation of motion:

$$\dot{\vec{\mu}} = 2\mu\vec{\mu} \times (\vec{E} \times \vec{V}). \tag{34}$$

Thus, if our magnetic moment is moving in the field of a charged line directed along the z-axis, the variation which μ_z experiences during the time it completes a circle of

* Prof. G.I. Opat of the University of Melbourne is currently investigating the feasibility of obtaining the proposed effect (private communication).

radius r is:

$$|\frac{\Delta\mu_z}{\mu_z}|\simeq|\frac{2\mu\lambda\Delta z}{2\pi r}|\simeq|\frac{S_{AB}}{2\pi}|\cdot|\frac{\Delta z}{r}|. \tag{35}$$

We infer from eq. (35) that the A-B effect is insensitive to the non abelian nature of the effective vector potential so long as the velocity parallel to the line of charges is small compared to the transverse velocity. Note incidentally that eq. (34) holds true also in classical mechanics, if $\frac{1}{2}\vec{\sigma}$ is replaced by the internal angular momentum of the moment, and Poisson brackets are used.

5 A Quantum Field Theoretic Model

The Lagrangian (16) was derived by postulating that the internal structure of the magnetic moment is consistent with special relativity (eq. 12). It would be instructive and reassuring to derive eq. (16) from quantum field theory. To this end, consider a $(2+1)$ dimensional Higgs system[4] (a relativistic super-conductor) described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + [(\partial_\mu + ieA_\mu)\phi]^*[(\partial^\mu + ieA^\mu)\phi] - y^2(|\phi|^2 - v^2)^2. \tag{36}$$

As is well known, the $U(1)$ gauge symmetry is spontaneously broken and the photon gets a mass $M = \sqrt{2}ev$. This system has a solution which describes a localized quantized magnetic fluxon[5] whose field strength decreases exponentially outside the core:

$$e\vec{A}_{c\ell}(\vec{r}-\vec{R}) = \frac{\hat{z}\times(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^2} + O(e^{-M|\vec{r}-\vec{R}|}). \tag{37}$$

($\vec{r}$ is a two-dimensional vector and $\hat{z}$ is the unit vector perpendicular to the plane). Suppose we add an external charge density J_0, so that

$$\mathcal{L} \to \mathcal{L} - \int A_0 J_0. \tag{38}$$

In order to find A_0 we may neglect all the excitations (photons of mass M and scalars yv) and keep only the collective center of mass coordinates $\vec{R}$. The charge density operator ρ of the system is:

$$\rho = -ie\phi^*(\dot{\phi} + ieA_0\phi) + h.c. \tag{39}$$

Substituting $\phi(\vec{r},t) = \phi_{c\ell}(\vec{r} - \vec{R}(t))$ and using the field equations:

$$\begin{aligned} -\nabla^2 A_0 &= \rho \\ -\nabla^2 \vec{A}_{c\ell} &= [(\nabla - ie\vec{A})\ \phi_{c\ell}]^* ie\phi_{c\ell} + h.c., \end{aligned} \tag{41}$$

we find:

$$\mathcal{L} \to \mathcal{L} - \dot{\vec{R}} \cdot \int (d\vec{r}) J_0(\vec{r}) \vec{A}(\vec{r} - \vec{R}), \tag{42}$$

which is precisely eq. (4). This is of course not surprising since it is simply a realization of the previous derivation. Note however that if the external charge is viewed as a source, the electric field it generated must be screened by the system so that the total electric field seen by the fluxon is $O(e^{-M|\vec{r}-\vec{R}|})$. It is remarkable that the only piece of the field $\vec{E}$ which actually enters the effective Lagrangian is the unscreened Coulomb field of the external charge. The consistency of this result is explained by remarking that the term $\vec{V} \cdot \vec{E} \times \vec{\mu}$ generates no force and is only effective in inducing an A-B phase. The particles of the medium are however quantized in the proper unit pertaining to the flux so that their share of the phase $= 2\pi N$ and hence irrelevant. This phenomenon may be summarized by the statement that the superconductor screens all the moments of $\vec{E}$ but does not screen the topological effect of $\exp i \oint d\vec{\ell} \cdot \vec{\mu} \times \vec{E}$. The above discussion suggests the possibility of looking for the effect on fluxons in two-dimensional superconductors.

An amusing and deep property of relativistic superconductors emerges when the A-B effect of the fluxons is considered in a new context. Suppose we create a coherent state of fluxons and designate the Schrödinger field of the fluxons by $\Psi(\vec{R})$. This field should now become classical and satisfy the classical Schrödinger equation (m is the fluxon mass):

$$[i\partial_t + \frac{1}{2m}(\nabla - ie\vec{\mu} \times \vec{E})^2]\Psi = 0. \tag{43}$$

We now arrive at a contradiction: on the one hand the quantum mechanical particle interpretation predicts that the phase of the field Ψ will change by S_{AB} when the wave moves around a charge; on the other hand equation (43) is a local classical equation, and is sensitive only to $\vec{\mu} \times \vec{E} \propto e^{-MR} \sim 0$ so that no phase is generated. We conclude that a coherent state of fluxons will necessarily destroy the superconductivity and restore Coulomb's law: the condensation of the charged field ϕ and the fluxon field Ψ are mutually exclusive.

The dynamical mechanism which prevents the generation of a coherent fluxon state is the impossibility of superposing fluxon solutions due to the non linearity of the field equations. This theorem was first proved by G. 't Hooft[6] by considering the commutation rules of the field Ψ and the phase operator $\exp ie \oint \vec{A} \cdot d\vec{\ell}$.

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