A NEW CHARACTERISTIC OF A QUANTUM SYSTEM BETWEEN TWO MEASUREMENTS - A "WEAK VALUE".

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ABSTRACT. New characteristics of a quantum system between two measurements - "weak values" are defined. It is proved that the "weak values" are outcomes of "weak measurements" which are standard measuring procedures with "weakened" coupling.

1. Introduction

Suppose we have performed the following set of successive measurements on a quantum system. At time t_1 we found $A=a$ (a is a nondegenerate eigenvalue). At time t_2 we accomplished the standard measurement procedure for the observable C but with "weakened" coupling with the measuring device. At time t_3 we found $B=b$ (b is also a nondegenerate eigenvalue). The effective free Hamiltonian for the system is zero. Then the outcome of the measurement of C yields the new surprising value which we call the *weak value*:

$$C_w = \frac{\langle B{=}b|\, C\, |A{=}a\rangle}{\langle B{=}b|A{=}a\rangle} \tag{1}$$

This is a general result: for *any* system under the above circumstances, with a weak enough measurement of C, the most probable outcome will be C_w. The weak value of C can differ widely from any eigenvalue of C. In particular, the real part ReC_w can be much bigger (smaller) than the maximum (minimum) eigenvalue of C. The weak value can yield any value: spin component of a spin-$\frac{1}{2}$ particle can be 100,[1] kinetic energy can be negative, the weak value can even be a complex number...

I hope you are still reading, although probably in compete disbelief: what does it mean when the outcome of measurement of a physical variable is complex?! We have to explain some "buts" of the above amazing story. (It still will remain amazing). Of course, the outcome of the standard measurement procedure, even with coupling weakened enough, will not yield a complex value. The highest probability of the outcome will be at the value Re C_w. More than this, usually the outcome of the "weakened" measurement will have very large uncertainty and it will not be suitable even to find Re C_w. However, an ensemble of identical systems which are *both preselected and postselected* will allow us to find Re C_w. Indeed, the uncertainty of the

M. Kafatos (ed.), Bell's Theorem, Quantum Theory and Conceptions of the Universe, 17–22.

measurement will be improved by the factor $1/\sqrt{N}$. We can find also Im C_w by an "almost" standard procedure: by using the same "weakened interaction" but with some other way of "looking" at the measuring device.

2. The Proof that the "Weak Measurements" Yield "Weak Values"

Let us remind ourselves of the standard von Neumann measuring procedure. The Hamiltonian describing the interaction with a measuring device is:

$$H = -g(t)\ q\ C\ , \tag{2}$$

where $g(t)$ is a normalized function which is nonzero only near the time of measurement, and q is a canonical variable of the measuring device with conjugate momentum π. After the interaction (2) is over, we can ascertain the value of C from the final value of π:

$$C = \pi_f - \pi_{in} \equiv \delta\pi\ . \tag{3}$$

Any precise measurement of C necessarily disturbs in an uncontrollable manner the values of observables which fail to commute with C. The interaction (2) can be "weakened" by preparing an initial state of the measuring device for which the probability of finding a large q is sufficiently small. We shall now prove that such a "weak measurement" of C performed on an ensemble of systems, which were preselected in a state $|\Psi_1\rangle$ and were postselected in a state $\langle\Psi_2|$, will yield an outcome which we call a *weak value* of C :

$$C_w \equiv \frac{\langle\Psi_2|\,C\,|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}\ . \tag{4}$$

For convenience we take the initial state of each measuring device to be a Gaussian centered around zero in both the q and the π representations, which has a small spread in q. To simplify the following proof we note that changing the time ordering between measurements of the π and the postselection measurement will not affect any of the results of the measurement. The state of each (chosen) measuring device after the postselection (up to a normalization factor) is given by the following wave function in the q representation:

$$\langle\Psi_2|\ e^{-i\int H dt}\ |\Psi_1\rangle\ e^{-\frac{q^2}{4\Delta^2}} = \sum_{n=0}^{\infty} \frac{(iq)^n}{n!} \langle\Psi_2|\ C^n\ |\Psi_1\rangle\ e^{-\frac{q^2}{4\Delta^2}} =$$

$$= \langle\Psi_2|\Psi_1\rangle \sum_{n=0}^{\infty} \frac{(iq)^n}{n!} (C^n)_w\ e^{-\frac{q^2}{4\Delta^2}}\ . \tag{5}$$

By taking Δ such that

$$\Delta^n\ |(C^n)_w - (C_w)^n| \ll 1 \qquad \text{for all } n, \tag{6}$$

we can replace in the sum (5) all factors $(C^n)_w$ by $(C_w)^n$, so as to obtain:

$$\langle\Psi_2| \, e^{-i\int H dt} \, |\Psi_1\rangle \, e^{-\frac{q^2}{4\Delta^2}} \cong \langle\Psi_2|\Psi_1\rangle \sum_{n=0}^{\infty} \frac{(iq)^n}{n!} (C_w)^n \, e^{-\frac{q^2}{4\Delta^2}} =$$

$$= \langle\Psi_2|\Psi_1\rangle \, e^{iq \frac{\langle\Psi_2|C|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}} \, e^{-\frac{q^2}{4\Delta^2}} \quad . \tag{7}$$

The last wave function in the π representation is approximately

$$e^{-\Delta^2 \left[\pi - \frac{\langle\Psi_2 | C | \Psi_1\rangle}{\langle\Psi_2 | \Psi_1\rangle} \right]^2} \quad . \tag{8}$$

The probability distribution of π is a Gaussian with spread $\Delta\pi = 1/(2\Delta)$ centered at $\pi = \mathrm{Re}\, C_w$, i.e. the outcome of the measurement is, indeed, C_w.

The weak value of C as defined by (4) may have, also, an imaginary part. $\mathrm{Im}\, C_w$ can be found by measuring the canonical variable q itself. Indeed, in the q representation the state of the measuring device (obtained by manipulating Eq.(7)) is:

$$e^{iq \mathrm{Re}\, C_w} \, e^{-\frac{1}{4\Delta^2} \left[q + 2\Delta^2 \mathrm{Im}\, C_w \right]^2} \quad . \tag{9}$$

Consequently, the probability distribution of q is a Gaussian with the same spread Δ centered at $q = 2\Delta^2 \ \mathrm{Im}\ C_w$. The uncertainty in π and q will not allow us to deduce $\mathrm{Re} C_w$ or $\mathrm{Im}\ C_w$ from a single measurement. However, performing the measurement on an ensemble of N systems will decrease the uncertainty of the outcome by the factor $1/\sqrt{N}$. Therefore, by taking N large enough $(1/(2\Delta\sqrt{N}) \ll \mathrm{Re}\, C_w, \ \mathrm{Im}\ C_w)$ we can measure the complex value of C_w with any precision.

3. The Experiment Which Measures a Weak Value of a Spin Component of a Spin-$\frac{1}{2}$ Particle

We shall now describe an experiment that measures the weak value of the z component of a spin-$\frac{1}{2}$ particle and yields an arbitrarily large outcome. A version of this experiment can, we believe, be performed in the laboratory.

We start with a beam of particles moving in the y direction with a well defined velocity. The particles are initially well localized in the xz plane and have their spins pointed in a direction $\hat{\xi}$. We choose $\hat{\xi}$ in the xz plane with an angle α between $\hat{\xi}$ and $\hat{x}$ (Fig.1). The prepared beam comes through a Stern-Gerlach device which measures the spin weakly in the z direction. The requirement of weakness is fulfilled by making the gradient of the magnetic field sufficiently small. The motion of the beam changes, therefore, only slightly. This weak measurement causes the spatial part of the wave function to change into a mixture of two slightly shifted functions in the p_z representation, correlated to the two values of σ_z. We then pass the particles

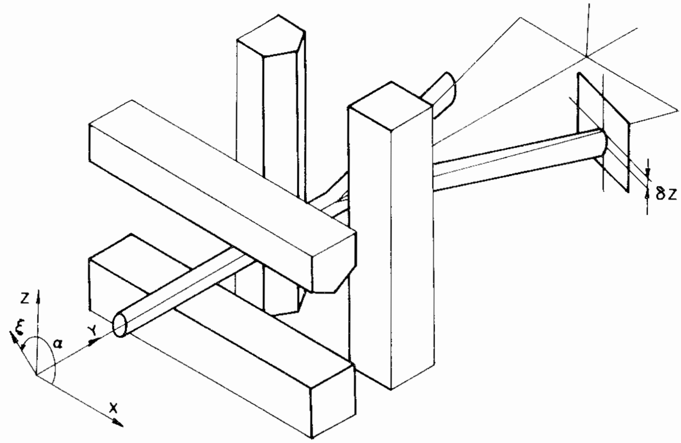

Figure 1. The experiment for measurement of weak value of the spin component of the spin-$\frac{1}{2}$ particle.

through another, normal, Stern-Gerlach device which splits them into two beams corresponding to the two values of σ_x. We keep only the beam with $\sigma_x = 1$, which continues to move freely towards a screen placed in front of it. The screen is placed sufficiently far, so that the displacement in the $\hat{z}$ direction due to the average momentum p_z, acquired during the above weak interaction, will be larger than the initial uncertainty Δz. On the screen we shall obtain a wide spot whose displacement in the direction $\hat{z}$ is measured. This displacement δz will yield the weak value of σ_z:

$$\sigma_{z\,w} = \frac{\langle \uparrow_x \mid \sigma_z \mid \uparrow_\xi \rangle}{\langle \uparrow_x \mid \uparrow_\xi \rangle} = \tan \frac{\alpha}{2} \ . \qquad (10)$$

The amplification of the displacement in the z direction (by the factor of $\tan\alpha/2$ beyond any "allowed" value) is caused by postselection measurement. But the latter interacts only in the x direction! Although the outcomes of "weak" measurements seem to contradict the laws of quantum mechanics, our approach never disputes the validity of the standard approach. However, using the standard formalism, the surprising result can be explained only by the following mathematical "miracle".

4. A Mathematical "Miracle"

Look on the Fig.2. Can you believe that the sum of all solid-line graphs yields the dashed-line graph (there is a scale factor 10^6 between the graphs)? The solid lines are all graphs of the same function $f(\pi)$ which are shifted and multiplied by different values. All shifts are in the range [1,-1] while the sum (dashed line) is practically the same Gaussian shifted by the value 3. The figure represents the following equality:

$$\sum_{n=0}^{N} c_n\, f(\pi + \frac{2n-N}{N}) \cong f(\pi - \frac{1}{\cos\,\alpha/2})\,, \tag{11}$$

where c_n are:

$$c_n = \left[\frac{\cos^2\frac{\alpha}{4}}{\cos\frac{\alpha}{2}}\right]^N \left[-\tan^2\frac{\alpha}{4}\right]^n \frac{N!}{n!(N-n)!}\;. \tag{12}$$

The parameters are: $\alpha = 70^0$, $N = 15$, $f(\pi) = e^{-\frac{\pi^2}{2.56}}$.

For large N the above formula is correct not only for this Gaussian, but for a large class of functions f. The coefficients c_n are the same for different functions $f(\pi)$! But the coefficients, of course, depend on α which parametrizes the shift of the sum. The requirement for the $f(\pi)$ is that its Fourier transform falls with the large q fast enough:

$$|\tilde{f}(q)| < e^{-c|q|}\;, \qquad c = \frac{1}{2}\tan^2\frac{\alpha}{2}\;. \tag{13}$$

Any function which fulfills this condition will behave like the one on Fig.2.

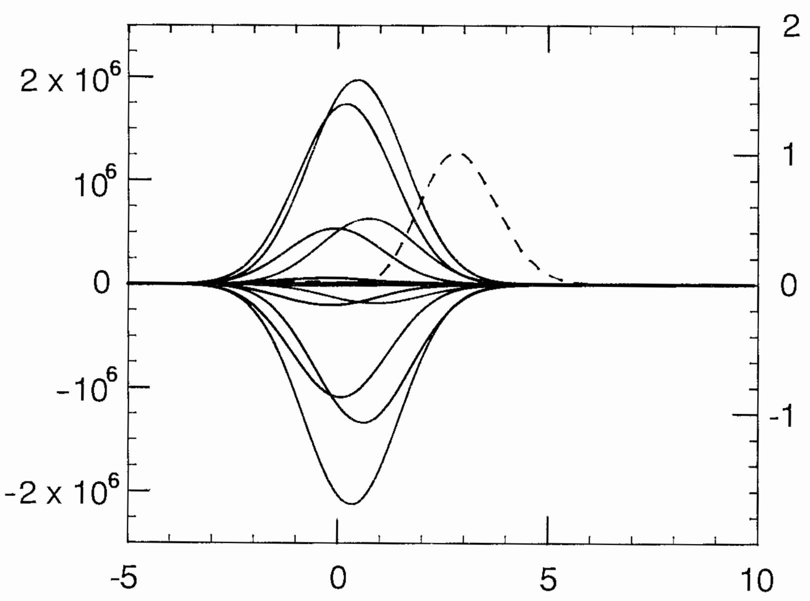

Figure 2. Sum of the Gaussians centered between -1 and 1 (solid lines, scale on the left) yields, approximately, a Gaussian centered at the value 3 (dashed line, scale on the right).

Eq.(11) represents the shift of the wave function in the π representation of the center of mass of the ensemble of measuring devices in the experiment described in Sec.3. (More precisely, the measurement of the spin here is in the direction which bisects the angle α. This is the direction of maximal amplification.) The width of the Gaussian $\Delta\pi$ represents the "weakness" of the measurement and N is the number of systems in the ensemble.

5. Conclusions

We discovered that a quantum system between two measurements can be described by novel characteristics: "*Weak values*". These weak values can be measured on ensembles which are both preselected and postselected. Although the weak value of C can differ widely from any "allowed" value of C, the standard measuring procedure changed only by "weakening" the interaction will yield the real part Re C_w and a small modification will give us Im C_w .

The requirement of "weakness" of the interaction which ensures obtaining the weak value is not an extraordinary one. In fact, most of experiments which are done in today's laboratories fulfill the "weakness" condition. Nobody has seen the "weak values" so far, because usually the experimentalists do not deal with ensembles which are both preselected and postselected. It seems, however, that such experiments can be done.[1,2] We believe that a set of Stern-Gerlach devices or their optical analogue can easily suit this purpose.

The importance of the weak values is that the effective value of C for *any* (weak enough) interaction with the systems in preselected and postselected ensembles is, in fact, the weak value C_w. We believe that the idea of weak measurement has a large potential for practical usage. Weak measurement performed on a both preselected and postselected ensemble can effectively amplify or "tune" any physical variable to a certain (even "forbidden") value.

References

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