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Is there a preferred canonical quantum gauge?

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The interaction between a long solenoid and a charged particle in the field free region outside it is studied treating both systems quantum mechanically. This leads to a paradox which suggests that when the electromagnetic field is quantized, there may be a preferred quantum gauge for the vector potential. This paradox is resolved by canonically quantizing the system in a different gauge in which the classical Lagrangian or Hamiltonian contains an acceleration dependent term.

It was shown by Aharonov and Bohm (AB) [1] that there is a phase shift in the interference of two coherent charged particle beams given by the phase factor

$$u_\gamma = \exp\left(-\frac{ie}{\hbar c} \oint_\gamma A_\mu \, \mathrm{d}x^\mu \right), \tag{1}$$

even when the particle beams are in a region in which the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$, where A_μ is the four-vector potential and γ is a closed curve that goes through the interfering beams. On the basis of this effect, Wu and Yang [2] stated that an intrinsic and complete description of the classical electromagnetic field is provided by the phase factor (1) [#1]. The field strength $F_{\mu\nu}$ underdescribes the electromagnetic field in a multiply connected region. But A_μ overdescribes the field because of its gauge freedom. Therefore, the AB effect demonstrates the reality of the gauge invariant holonomy transformation (1) and not A_μ.

In this Letter we consider an effect involving the *quantized* electromagnetic field which raises the question of whether there is a preferred gauge for canonically quantizing the electromagnetic field, which would give reality to A_μ. We show, however, how to describe this effect in different gauges. But in doing so, we quantize a system for which the Lagrangian contains a term that depends on the acceleration of a canonical coordinate, by imposing canonical commutation relations which are different from the usual ones.

Throughout this Letter we neglect terms of $\mathrm{O}(v^2/c^2)$. Consider the example of an infinitely long charged circular cylinder rotating about its fixed axis, which produces a magnetic field analogous to a long solenoid, and a particle with charge e outside it. For simplicity, the electrostatic interaction is removed by putting a uniform stationary line charge inside the cylinder with its linear charge density opposite to that of the cylinder so that the electromagnetic field strength is zero everywhere outside the cylinder. This model, which was studied by Peshkin, Talmi and Tassie [3], incorporates the dynamics of the source of the magnetic field as well as the particle, and we shall later quantize the degrees of freedom of both.

Let (r, θ, z) be the cylindrical coordinates of the particle with the z-axis along the axis of the cylinder and $\dot\beta$ is the angular velocity of the cylinder whose moment of inertia is I, where the overdot denotes differentiation with respect to time. Throughout this

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[#1] For this statement, e should be the charge of the smallest unit of charge, which at present is believed to be one third of the electron charge, and the charges of all physical systems are assumed to be an integer multiple of this fundamental unit of charge.

Letter we shall neglect the radiation due to any angular acceleration $\ddot{\beta}$. The Lagrangian for the combined system is

$$L=\tfrac{1}{2}m(\dot{r}^2+\dot{z}^2+r^2\dot{\theta}^2)+\tfrac{1}{2}I\dot{\beta}^2+\frac{ek}{c}\dot{\beta}\dot{\theta}\,. \tag{2}$$

The interaction term in (2) can be justified on the grounds that it gives the correct equations of motion for θ and β as determined by the forces acting on the system. It can also be expressed as $(e/c)\boldsymbol{A}\cdot\boldsymbol{v}$ where $\boldsymbol{A}=(k/r)\dot{\beta}\boldsymbol{e}_\theta$, with $\boldsymbol{e}_\theta$ being a unit vector in the direction of increasing θ. Then $\mathrm{div}\,\boldsymbol{A}=0$ and $\mathrm{curl}\,\boldsymbol{A}=\boldsymbol{0}$ outside the cylinder. But a gauge transformation can be made by adding a total time derivative of a suitable function to L which does not affect the equations of motion. The canonical momenta

$$p_\theta=mr^2\dot{\theta}+\frac{ek}{c}\dot{\beta}\,,\quad p_\beta=\frac{ek}{c}\dot{\theta}+I\dot{\beta}\,,\quad p_z=m\dot{z}$$

are constants of motion. Therefore, eliminating $\dot{\beta}$,

$$S\equiv[1-\epsilon(r)]mr^2\dot{\theta}=p_\theta-\frac{ek}{cI}p_\beta \tag{3}$$

is also a constant of motion, where $\epsilon(r)=e^2k^2/mc^2Ir^2$. The Hamiltonian [3]

$$H=\frac{1}{2m}(p_r^2+p_z^2)+\frac{1}{2I}p_\beta^2+\frac{1}{2mr^2(1-\epsilon)}\left(p_\theta-\frac{ek}{cI}p_\beta\right)^2\,. \tag{4}$$

Clearly,

$$\alpha\equiv\theta+\frac{cI}{ek}\beta-\frac{c}{ek}p_\beta t$$

is a constant of motion and the Poisson bracket

$$\{S,\alpha\}=0\,. \tag{5}$$

Now, take the limit of large I. The initial conditions at time t_0 are chosen so that α is finite, i.e. $(c/ek)\beta$ is $\mathrm{O}(1/I)$. Then, because α is conserved, it remains finite. In this limit, since $r\neq0$ for the particle outside the cylinder, $\epsilon\to0$. Then S is the kinetic angular momentum of the particle, $\dot{\beta}\to p_\beta/I$ is a constant, and $\boldsymbol{A}\to(k/Ir)p_\beta\boldsymbol{e}_\theta$.

We now quantize this theory in the usual way. Sometimes, for the sake of emphasis or if the meaning is not clear from the context, we shall notationally distinguish a quantum observable from the corresponding classical observable by a caret. Then (5) is replaced by

$$[\hat{S},\hat{\alpha}]=0\,, \tag{6}$$

and the Heisenberg observables $\hat{S}$ and $\hat{\alpha}$ are constants of motion. Let $|\Psi\rangle$ be the state of the combined system. A particle is said to be confined to a certain region if $\langle r|\Psi\rangle$, $\langle\theta|\Psi\rangle$ and $\langle z|\Psi\rangle$ can be non-zero only inside this region. The set of states for which the particle is outside the cylinder forms a Hilbert space $\mathscr{H}$. When the Hamiltonian acts on $\mathscr{H}$, $\hat{\boldsymbol{A}}\equiv(k/Ir)\hat{p}_\beta\boldsymbol{e}_\theta$, which is minimally coupled to the particle, may be regarded as the quantized vector potential experienced by the particle. This has been quantized in the gauge in which $\mathrm{div}\,\boldsymbol{A}=0$. The requirement that A_μ is zero at infinity then uniquely determines A_0 as a function of the charge density ρ if charges are present. Therefore our quantized potentials satisfy, in the present low energy limit,

$$\mathrm{div}\,\hat{\boldsymbol{A}}=0\quad\text{and}\quad\hat{A}_0=\int\frac{\hat{\rho}(x',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,. \tag{7}$$

Since in the present case A_0 experienced by the particle is 0, the Coulomb gauge condition (7) is consistent with the Lorentz gauge condition $\partial_\mu\hat{A}^\mu=0$, which has the advantage of being Lorentz covariant.

But a different gauge can be chosen in which the vector potential is $\boldsymbol{A}'(\boldsymbol{x},t)=\boldsymbol{A}(\boldsymbol{x},t)-\nabla\Lambda(\boldsymbol{x},t)$ is zero in a simply connected region U outside the cylinder. The corresponding quantized potential $\hat{\boldsymbol{A}}'(\boldsymbol{x},t)$ must also vanish in U. Then S is replaced by $S'=p_\theta-(e/c)A'_\theta$. In U, the vanishing of A'_θ implies $[\hat{S}',\hat{\alpha}]=-i\hbar$ whose classical limit is $\{S,\alpha\}=1$, if the usual canonical commutation relations are assumed. But this is in conflict with (5), if α has the same physical meaning in both gauges.

The physical meaning of (5) or (6) is as follows: the time dependent magnetic field due to the particle inside the cylinder exerts a torque on the cylinder by means of the corresponding electromotive force. Therefore θ and β become correlated which is the meaning of α being a constant of motion. But the velocity of the charged particle is constant for large I because no forces act on it. Therefore it should be possible to specify, independently, α and the kinetic angular momentum S. The fact that this physical requirement does not seem to be satisfied in some gauges, as discussed above, leads to a paradox, because we know that the electromagnetic theory can be quantized using the Feynman path integral formalism in a manner which treats all gauges on an equal footing since the action is the same in all gauges.

Of course, the theory is invariant under a c-gauge transformation of the form $\hat{A}''_\mu(\boldsymbol{x},t)=\hat{A}_\mu(\boldsymbol{x},t)-\partial_\mu\Lambda(\boldsymbol{x},t)$, where Λ is a real valued function of space-time. The question raised here is whether the theory is invariant under a q-gauge transformation for which Λ depends on observables that do not commute with the canonical coordinates. The above paradox suggests that we are justified in quantizing the theory by the usual canonical commutation relations in the Coulomb gauge but not in other gauges, in general.

To understand this effect in other gauges, and to resolve this paradox, we consider a specific gauge namely an axial gauge in which $A'_x=0$ everywhere. The vector potential in this gauge $\boldsymbol{A}'=\boldsymbol{0}$ everywhere in the simply connected region U that is now defined as follows: suppose that the cylinder has radius a and its axis is at $x=0$, $y=0$; then U is the region *outside* the cylinder excluding the region between $y=a$ and $y=-a$ with $x<0$. In the latter region, $A'_x=0=A'_z$ and

$$A'_y(y)=4\frac{k}{I}\frac{(a^2-y^2)^{1/2}}{a^2}p_\beta .$$

But in this gauge the scalar potential in U is $A'_0=(k/c)\ddot{\beta}\theta$. This can be derived by eliminating Λ in the gauge transformations: $\boldsymbol{A}'=\boldsymbol{A}-\nabla\Lambda=\boldsymbol{0}$ and $A'_0=-c^{-1}\partial\Lambda/\partial t$. Physically, A'_0 is the potential for the electric field produced when the cylinder is given an angular acceleration $\ddot{\beta}$. Therefore the corresponding Lagrangian

$$L=\tfrac{1}{2}m(\dot{r}^2+\dot{z}^2+r^2\dot{\theta}^2)+\tfrac{1}{2}I\dot{\beta}^2-\frac{ek}{c}\ddot{\beta}\theta \tag{8}$$

contains a term proportional to the *acceleration* $\ddot{\beta}$. Therefore, if β is to be treated as an independent dynamical degree of freedom, instead of a fixed externally specified parameter, we cannot canonically quantize the theory by means of the usual procedure.

In order to see how this theory can be quantized in the present gauge, we perform a unitary transformation V from the Coulomb gauge to the present gauge which in the region U is

$$V=\exp\left(-\mathrm{i}\frac{ek}{c\hbar I}\theta p_\beta\right).$$

This has no explicit time dependence and therefore it does not introduce an A_0. The new Hamiltonian in this region is

$$H'=VHV^{-1}=\frac{1}{2m}(p_r^2+p_z^2)+\frac{1}{2I}p_{\beta'}^2+\frac{1}{2mr^2(1-\epsilon)}p_\theta^2 , \tag{9}$$

where $\beta'=\beta-(ek/cI)\theta$ is the transform of β in this region, whereas θ is transformed to $\theta'=\theta$. Then $p_{\beta'}=-\mathrm{i}\hbar\partial/\partial\beta'=p_\beta$ by the chain rule. Also,

$$[p_\theta,\beta']=\frac{\mathrm{i}\hbar ek}{cI}, \tag{10}$$

with the other commutation relations which are independent of (10) being the same as the usual ones.

We now take the angular coordinate of the cylinder to be β' and not β, while θ still retains its original meaning. Then (9) is a new Hamiltonian which is no longer related to the old Hamiltonian (4) by a unitary transformation, because the commutation relations have been changed. It corresponds to a gauge in which there is no vector potential in U. Then p_θ, the canonical momentum conjugate to θ, has the same physical meaning as what was denoted by S in the Coulomb gauge. It follows that we should quantize in this gauge by imposing the commutation relation (10). This is unlike in the usual canonical quantization scheme, which we used in the Coulomb gauge, in which the canonical momentum of the particle commutes with the canonical coordinate of the cylinder. This difference is due to the fact that there is an acceleration dependent term in the Lagrangian (8), whose effect in the quantum theory is obtained from (10). The non-commutativity of p_θ and β' in (10) can also be physically understood as being due to the fact that a measurement of β' results in the angular acceleration $\ddot{\beta}'$ of the cylinder which produces an electric field which changes p_θ which has the meaning of the kinetic angular momentum of the particle. Therefore, p_θ cannot commute with β' which is consistent with (10). With this new commutation relation and the new physical meanings assigned to p_θ and β', all physical consequences are the same in the present gauge and the Coulomb gauge.

We shall now justify the above quantization, which we did using physical arguments, by means of a rigorous procedure for quantizing a Lagrangian with an acceleration dependent term. First substitute $\omega\equiv\dot{\beta}$ and modify the Lagrangian (8) by adding to it the term $\lambda(\dot{\beta}-\omega)$. The new Lagrangian in U,

$$\bar{L}=\tfrac{1}{2}m(\dot{r}^2+\dot{z}^2+r^2\dot{\theta}^2)$$
$$+\tfrac{1}{2}I\omega^2-\frac{ek}{c}\dot{\omega}\theta+\lambda(\dot{\beta}-\omega)\,, \qquad (11)$$

gives the same equations of motion as the old Lagrangian on varying r, z, θ, ω and the Lagrange multiplier λ. Therefore it represents the same physics. But it has the advantage that it does not contain any acceleration. Here $\lambda=p_\beta$ and the Hamiltonian is

$$\bar{H}=\frac{1}{2m}(p_r^2+p_z^2)+\frac{p_\theta^2}{2mr^2}+p_\beta\omega-\tfrac{1}{2}I\omega^2\,, \qquad (12)$$

where the new canonical momenta are obtained by taking the derivatives of $\bar{L}$ with respect to the corresponding velocities.

But p_ω obtained this way satisfies the primary constraint equation

$$\chi^1\equiv p_\omega+\frac{ek}{c}\theta\approx 0\,, \qquad (13)$$

where $\approx$ represents weak equality meaning that χ^1 should be set to zero in the equations of motion *after* the Poisson brackets are all evaluated. We follow now the general procedure given by Dirac [4] for quantizing systems with constraints, i.e. relations among the canonical coordinates and momenta. The total Hamiltonian is of the form

$$H''=\bar{H}+u\chi^1\,, \qquad (14)$$

where the coefficient u is to be determined. Also, (13) must be valid for all times which implies the constraint

$$\chi^2\equiv\dot{\chi}^1=\{\chi^1,H''\}=-p_\beta+I\omega+\frac{ek}{mc}\frac{p_\theta}{r^2}\approx 0\,. \qquad (15)$$

Now the requirement that $\dot{\chi}^2\approx 0$ can be used to obtain u.

An observable whole Poisson bracket with each constraint weakly vanishes is said to be first class. Otherwise it is called second class. Here χ^1 and χ^2 are second class because

$$\{\chi^1,\chi^2\}=-I(1-\epsilon)\,. \qquad (16)$$

In the present approximation of $\epsilon\to 0$, which corresponds to large I or large m, $\{\chi^1,\chi^2\}=-I$. If the constraints were all first class then the theory can be quantized by replacing the Poisson brackets between the canonical variables by the usual commutators. But since they are second class, we modify the Poisson bracket to the Dirac bracket, which is defined for any two observables ξ and ζ to be

$$\{\xi,\zeta\}_D\equiv\{\xi,\zeta\}-\{\xi,\chi^a\}c_{ab}\{\chi^b,\zeta\}\,, \qquad (17)$$

where c_{ab} is the inverse of the matrix $c^{ab}\equiv\{\chi^a,\chi^b\}$, a, $b=1$, 2, i.e.

$$c_{ab}c^{bc}=\delta_a{}^c\,,$$

and the summation convention is being used. The Dirac bracket of χ^a with any observable is zero. Therefore we can now put

$$\chi^a=0,\quad a=1,2\,, \qquad (18)$$

before working out the Poisson brackets, i.e. as strong equations.

On using (18) the Hamiltonian (14) becomes

$$H''=\frac{1}{2m}(p_r^2+p_z^2)+\frac{1}{2I}p_\beta^2+\frac{1-\epsilon}{2mr^2}p_\theta^2\,. \qquad (19)$$

Since $c_{11}=c_{22}=0$ and $c_{12}=-c_{21}=1/I(1-\epsilon)$, we easily compute

$$\{p_\theta,\beta\}_D=\frac{ek}{cI(1-\epsilon)},\quad \{p_\theta,\theta\}_D=-\frac{1}{1-\epsilon}\,,$$

$$\{p_\theta,p_r\}_D=-\frac{2\epsilon}{(1-\epsilon)r}p_\theta\,, \qquad (20)$$

while the remaining Dirac brackets between pairs of canonical variables in (19) that are independent of (20) are the same as the Poisson brackets. It can be shown that the equations of motion expressed in terms of r, θ, z, β and their time derivatives which are now obtained from (19) using the Dirac brackets, instead of the Poisson brackets, are equivalent to the equations of motion obtained from (2) or (4), to all orders in ϵ. This confirms that (19) and (20) in the present gauge represents the same physical situation as (2) or (4) in the Coulomb gauge.

This theory is now quantized by regarding the observables as operators acting on the Hilbert space of states, and replacing the Dirac brackets by the corresponding commutators as in the usual prescription for replacing Poisson brackets.

Returning to the classical theory, on using (20), we find $\dot{\theta}=p_\theta/mr^2$ for the Hamiltonian (19). But for the Hamiltonian (9), on using the usual Poisson bracket relation between θ and p_θ, we find $\dot{\theta}=p_\theta/(1-\epsilon)mr^2$. Hence to go from (19) to (9) we should perform the transformation

$$p_\theta=\frac{p'_\theta}{1-\epsilon(r)}\,. \tag{21}$$

This is not a canonical transformation except of course in the limit $\epsilon\to 0$. However, if we take p_θ and p'_θ to have the physical meanings determined by $\dot{\theta}$, as mentioned above, then under the transformation (21) the physics will not change. On substituting (21) in (19) and (20), we obtain (9) and the associated bracket relations on dropping the primes:

$$\{p_\theta,\beta\}_{\rm D}=\frac{ek}{cI}\,,\quad \{p_\theta,\theta\}_{\rm D}=-1\,,$$
$$\{p_\theta,p_r\}_{\rm D}=0\,. \tag{22}$$

In the quantum theory the latter relations are replaced by commutators which include (10). Thus we have justified the non-commutativity of p_θ and β in a quantum gauge in which $\hat{\boldsymbol{A}}=\mathbf{0}$ in U, assumed earlier on physical grounds. Also, the theory represented by (9) and (10) was shown to be physically equivalent to (4) and its associated canonical commutation relations in the Coulomb gauge. This completes the cycle which we began in the Coulomb gauge, showing again that the physics is the same in all gauges considered here.

However, we find that a representation of observables which obey the commutation relations corresponding to (20) in the Hilbert space of *wave functions* is the same as the representation we would have written down in the Coulomb gauge. Hence the Coulomb gauge is preferred in the sense that it simplifies the mathematics by avoiding the tedious procedure used above for quantizing in a different gauge, and it gives directly the Schrödinger representation which is used for space–time description of quantum theory. But the above procedure, which can be extended to other gauges, shows that the theory can be quantized in different gauges to obtain the same physics.

In conclusion, the present work shows an interesting connection between the role of the vector potential and the canonical commutation relations in quantum theory. This work can be generalized to a non-Abelian gauge field. Here also, a gauge transformation from the Coulomb gauge will in general result in a change in the canonical commutation relations. This will be treated in a future work.

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