Proc. Int. Symp. Foundations of
Quantum Mechanics, Tokyo, 1983, pp. 190–194

Quantum Frames of Reference*

Tirza KAUFHERR and Yakir AHARONOV†

102, Usishkin Street Tel, Aviv 62031, Israel
†*Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel*
and
†*Department of Physics, University of South Carolina, Columbia. South Carolina 29208, USA*

The present paper addresses itself to the problem of finite mass quantum frames of reference. We show that with the help of suitable geometrical potentials it is possible to construct a theory that takes account of the quantum disturbances of the reference frame in a consistent manner. We thus obtain a description of a physical system relative to reference frames described by the most general quantum states, within the framework of non relativistic theory. Our formalism satisfies an equivalence principle which presents an extension of the classical equivalence principle to quantum theory. It also augments the usual continuous Galilean symmetry with additional discrete symmetries corresponding to the transformation from one physical reference frame to another.

§1. Introduction

The present paper addresses itself to the problem of finite mass quantum reference frames. It was Eddington who first observed both the necessity and also the difficulty of describing a physical system in quantum theory relative to a finite mass reference frame. Since then it has been thought that the uncontrolled quantum fluctuations of the reference frame render such a description impossible.

We show that with the help of suitable geometrical potentials it is possible to construct a theory that takes account of these quantum fluctuations in a consistent manner, thus obtaining a description of a physical system relative to reference frames described by the most general quantum states, within the framework of non relativistic theory. Our formalism satisfies an equivalence principle which presents an extension of the classical equivalence principle to quantum theory. It is hoped that this approach, once extended to relativistic theory, will contribute towards the solution of problems associated with the quantization of general relativity.

§2. The Paradox of the Finite Mass Quantum Reference Frame

We shall acquaint the reader with the concept of a finite mass reference frame by means of a paradox. This will be the starting point of our investigation.

Given are two reference frames O_1 and O_2, and a single particle, Q. The reference frames will each be thought of as finite mass laboratories containing clocks, rulers, *etc.*, all of which are rigidly attached to the walls of the laboratory. Thus, a reference frame will here be represented by a single degree of freedom, whose mass is finite, and whose center of mass defines the origin of the reference frame. Consequently, these reference frames may be put into a well defined quantum state relative to one another, which would not be possible if the mass were infinite.

Consider now the set of all possible experiments within one given reference frame. With regard to this, it is necessary to make the following statement: That it should be impossible, by means of an experiment performed within one given reference frame, to discover its state of motion. This requirement is familiar from Galilean invariance and from the special and the general theories of relativity. But here it acquires richer meaning, for in the present case the states in question are quantum states. Thus, O_1 may be in an eigenstate of position or of momentum relative to the external frame. The above requirement then means, that it should be impossible, for an experimenter

* Presented by T. Kaufherr.

active within O_1, to distinguish between these states.

For the sake of simplicity, we shall, in what follows, restrict the discussion to the one dimensional case.

Notation:
O_1-*internal* frame-unscripted variables.
Mass: M_1.
O_2-*external* frame-upper script *ext*.
Mass: M_2.
Q-particle. Mass: m.

We shall, in other words, demand that

$$[x_{\text{particle}}, V^{\text{ext}}_{\text{CM}_1}]=0,$$

where

$$V^{\text{ext}}_{\text{CM}_1}=\frac{M_1 V^{\text{ext}}_{\text{O}_1}+mV^{\text{ext}}_{\text{particle}}}{M_1+m},$$

and further, similar conditions.* But the internal observer can measure the velocities of O_2 and of the particle. Thus, the velocity seems to him to be

$$\frac{M_1(-V_{\text{O}_2})+m(V_{\text{particle}}-V_{\text{O}_2})}{M_1+m},$$

and he would find

$$\left[x_{\text{particle}}, \frac{M_1(-V_{\text{O}_2})+m(V_{\text{particle}}-V_{\text{O}_2})}{M_1+m}\right]$$
$$=\frac{i\hbar}{M_1+m} \quad (\neq 0 \text{ unless } M_1=\infty).$$

It is possible to resolve this paradox by assuming the existence, in the internal frame, of a vector potential A such

$$mV_{\text{particle}}=P_{\text{particle}}+mA.$$

But as none of the two reference frames should be preferred, also

$$mV^{\text{ext}}_{\text{particle}}=P^{\text{ext}}_{\text{particle}}+mA^{\text{ext}}$$

and where, in particular A^{ext} is the same function of variables in O_2 as is A of variables in O_1.

The solution that follows is

$$V_{\text{particle}}=\frac{P_{\text{particle}}}{m}+\frac{P_{\text{particle}}+P_{\text{O}_2}}{M_1}+C$$

$$V^{\text{ext}}_{\text{particle}}=\frac{P^{\text{ext}}_{\text{particle}}}{m}+\frac{P^{\text{ext}}_{\text{particle}}+P_{\text{O}_1}}{M_2}+C'.$$

The necessity of describing space-time in terms of relative variables and of including the reference frame itself in the system under consideration which then follows, was noticed early in the history of quantum theory, but has remained an open question since.[2] Eddington was among the first who realized this point and who stressed its importance.[1]

Beyond this, the question that is being raised by our paradox concerns the consistency of the quantum description. For the paradox seems to suggest that quantum theory is in principle unable to encompass the whole universe in its description, but it cannot even be made to consistently support the notion of a finite mass observer and that consequently it needs a classical-type theory to augment it.

§3. Finite Mass Quantum Frames of Reference

We shall now describe in detail the covariant solution for N particles relative to a finite mass reference frame in one dimension. Following this we shall give the solutions for the geometrical potential in the 2 and 3 dimensional cases, and point out some of the interesting features that appear.*

Let us then consider N free particles in one dimension. Their masses are

$$m_0, \cdots, m_N.$$

We assume the existence of an 'absolute', infinite mass, reference frame. This 'absolute' frame will serve as an auxiliary frame, and will be completely abandoned later on. The positions and momenta over the absolute frame are

$$x_0, \cdots, x_N$$
$$P_0, \cdots, P_N$$

and the canonical commutation relations are assumed.

$$[x_i, P_j]=i\delta_{ij}, \quad i,j=0,\cdots, N.$$

The hamiltonian over the 'absolute' frame is

$$H=\sum_{n=0}^{N}\frac{P_n^2}{2m_n}.$$

* For the complete discussion of this paradox, see ref. 4.

* The full solution of the 2 and 3 dimensional cases is given in ref. 4.

Because this commutes with the total momentum, we can, without loss of generality, assume that the system is in an eigenstate of $P_T=0$, P_T denoting the total momentum.

Let us now choose particle Q to define the origin of a 'relative' frame. In this capacity it resembles the observers O_1 and O_2 mentioned in §2. We further define a new set of canonical coordinates and momenta such

$$q_0=x_0, \quad \pi_0=P_T,$$

$$q_n=x_n-x_0, \quad \pi_n=P_n,$$

clearly,

$$[q_i, \pi_j]=\mathrm{i}\delta_{ij}, \quad i,j=0,\cdots,N$$

is satisfied.

The characteristic feature about this choice is that the coordinates have been defined as relative variables, in particular, as distances from particle Q.

The covariant hamiltonian defined in the relative frame is

$$H'(\pi_n)=\sum_{n=1}^{N}\frac{\left(\pi_n+m_n\frac{\Pi}{m_0}\right)^2}{2m_n}-\frac{M\Pi^2}{2m_0^2},$$

where

$$\Pi=\sum_{n=1}^{N}\pi_n,$$

and M is the total mass.

An important characteristic of our formalism is its generality. To see this, let us include in the absolute hamiltonian an interaction such as

$$H=\sum_{n=0}^{N}\frac{P_n^2}{2m_n}+V(x_e-x_0).$$

Clearly, the definitions of the π's and q's will follow unchanged. Only that in the present case the relative frame is a non inertial frame, for particle Q, the reference particle, is accelerating. Thus, our definition of the relative frame allows for the most general type of motion of the reference frame.

Let us briefly explain the meaning of the vector potential. In the case where the relative frame is a non inertial frame, our geometrical vector potential gives rise to the inertial, Coriolis and centrifugal type forces. But as we have seen, the potential appears in the hamiltonian even if no forces are acting. In this case, it represents the velocity of the reference frame and, in particular its quantum spread.

The velocities in the relative frame are given by

$$V_e=\dot{q}_e=\frac{\partial H'}{\partial\pi_e}=\frac{\pi_e}{m_e}+\frac{\Pi}{m_0}$$

which conforms with the result of the previous section.

The unitary transformation that transforms to another relative frame where particle 1, namely, serves as reference particle is given by

$$U=P_1\exp\left(\mathrm{i}\sum_{2}^{N}\pi_n q_1\right)\exp(-\mathrm{i}\pi_0 q_1)$$

where P_1 is the parity operator for particle 1 and $q_n=Uq_nU^\dagger$, $\pi_n=U\pi_nU^\dagger$.

Let us now briefly consider the 2 and 3 dimensional cases. In two dimensions at least two degrees of freedom are needed, so as to define both origin and a direction in the plane.

The hamiltonian is

$$\bar{H}=\sum_{2}^{N}\frac{P_n^2}{2m_n}+\frac{\Pi^2}{2M}+\frac{1}{2\mu}\left(P_x+\frac{yL_z}{x^2+y^2}\right)^2+\frac{1}{2\mu}\left(P_y-\frac{xL_z}{x^2+y^2}\right)^2 \tag{1}$$

where

$$H'(q_n,\pi_n)=U\bar{H}(x_n,P_n)U^\dagger,$$

i.e., H' relates to $\bar{H}$ simply by change of names of the respective variables, and

$$\Pi=\sum_{2}^{N}P_n, \quad L_z=\sum_{n\neq \mathrm{rel}}L_{zn}.$$

The variables x, y, P_x, P_y above belong to a reference particle (degree of freedom) named "rel" whose mass is μ. Inspecting (1) we observe that the vector potential appearing in the kinetic energy term of this reference particle corresponds to that of a singular line of flux along the z axis. Because of the quantization of angular momentum, this flux introduces no observable effect.

At least 3 degrees of freedom are needed to define a 3-dimensional physical reference frame. The hamiltonian is

$$\bar{H}=\sum_{u\neq \mathrm{rel}}\frac{P_n^2}{2m_n}+\frac{(P_x+A_x)^2}{2\mu}+\frac{(P_y+A_y)^2}{2\mu}+\frac{(P_z+A_z)^2}{2\mu}, \tag{2a}$$

where the notation used is similar to the above. The solution for the vector potential is

$$\begin{aligned}
A_x&=\frac{1}{r}\left\{-x\left(\frac{zL_y}{r\sqrt{x^2+y^2}}\right)+y\left(\frac{-L_x}{\sqrt{x^2+y^2}}+\frac{zL_z}{x^2+y^2}\right)\right\},\\
A_y&=\frac{-1}{r}\left\{y\left(\frac{zL_z}{r\sqrt{x^2+y^2}}\right)+x\left(\frac{-L_x}{\sqrt{x^2+y^2}}+\frac{zL_z}{x^2+y^2}\right)\right\},\\
A_z&=\frac{\sqrt{x^2+y^2}}{r^2}L_y.
\end{aligned} \tag{2b}$$

The fields are no more Abelian. The Coriolis type fields vanish, *i.e.*,

$$B_k=\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}+\mathrm{i}[A_i, A_j]\equiv 0$$

accepts on a set of points of measure zero. If one calculates the Abelian part of the field, *i.e.*, $\mathrm{B}_k\equiv\partial A_j/\partial x_i-\partial A_i/\partial x_j$, one finds that the contribution that is proportional to L_z behaves like a Coulomb field.

$$\begin{aligned}
\bar{B}_x(L_z)&=\frac{x}{r^3}L_z,\\
\bar{B}_y(L_z)&=\frac{y}{r^3}L_z,\\
\bar{B}_z&=\frac{z}{r^3}L_z.
\end{aligned} \tag{3}$$

Calculating the line integral of A around a small circle of radius r_{xy} surrounding the z-axis, one find

$$\lim_{r_{xy}\to 0}\oint A\cdot \mathrm{d}l=\begin{cases}-2\pi L_z & z>0\\ 2\pi L_z & z<0.\end{cases} \tag{4}$$

Combining (3) and (4) we find that the potential has a singularity on the z axis that corresponds to two monopoles[3] at the origin, their strings lying along the $+z$, $-z$ axis, respectively. Again, since the angular momentum L_z is quantized, there is no observable effect due to these singular fluxes.

§4. Conclusions

The problem of describing space-time in terms of relative variables which entails the inclusion of the reference frame itself in the system under consideration, has been solved. One of the first to propose this problem was Eddington.

In this solution, a canonical description of N particles relative to a finite mass reference frame is given in closed form, by means of a covariant hamiltonian, together with the transformation law between different reference frames.

Besides solving the consistency problem, the solution also constitutes an extension of the classical equivalence principle to quantum theory, since a covariant description relative to a quantum reference frame, whose motion may be of the most general kind, is given. So far, non relativistic theory has been treated. We intend to extend the treatment to relativistic theory, in the hope that this approach will contribute towards the solution of problems associated with the quantization of general relativity.

References

1) A. S. Eddington: *The Nature of the Physical World* (The McMillan Company, New York, Cambridge University Press, Cambridge, England, 1929) p. 220; A. S. Eddington: *Fundamental Theory* (Cambridge University Press, Cambridge, England, 1946).

2) A. Einstein and L. Infeld: *The Evolution of the Physical World* (Simon and Schuster, New York, 1938) p. 220; R. P. Feynman, R. B. Leighton and M. Sands: *The Feynman Lectures on Physics* (Addison-Wesley Publishing Co., Reading, Mass., 1964) Vol. 2, §42–1–§42–14; N. Bohr: *Discussions with Einstein. Einstein, Philosopher, Scientist,* ed. P. A. Schilpp (The Library of Living Philosophers, 1949) Vol. 1, p. 224; Y. Aharonov and G. Carmi: Found. Phys. **3** (1973) 493; Y. Aharonov and G. Carmi: Found. Phys. **4** (1974) 75; E. Lubkin: Ann. Phys. (N.Y.) **56** (1970) 69; C. A. Mead: Chem. Phys. (Netherlands) **49** (1980) 1.

3) P. A. M. Dirac: Proc. Roy. Soc. A **133** (1931) 60; Phys. Rev. **74** (1948) 817; J. D. Jackson: *Classical Electrodynamics* (John Wiley and Sons Inc., New York, 1975) p. 251; C. N. Yang and R. L. Mills: Phys. Rev. **1** (1954) 191; G. 't Hooft: Nucl. Phys. B **79** (1976) 276.
4) Y. Aharonov and T. Kaufherr: to be published in Phys. Rev. D (1983).

* * *

M. Jammer: Did you consider the relation between your results and the result which arises out of Araki's condition that a measurement of any of the observables, *e.g.*, m_1 in system O_1 discussed in your representation can be only an approximation as long as the masses involved are not sufficiently large?

T. Kaufherr: Since we discuss the measurement of relative quantities only, such a difficulty as referred to in the question does not arise.

M. Peshkin: How did you make any progress for rotating frames?

T. Kaufherr: The 2- and 3-dimensional cases have been fully solved. The relation is naturally included in the formalism in these cases.

A. Zeilinger: Should not a reference frame consist of at least as many objects as the dimension of your space?

T. Kaufherr: Yes.