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Superluminal tunnelling times as weak values

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Abstract. We consider the tunnelling particle as a pre- and post-selected system and prove that the tunnelling time is the expectation value of the position of a 'clock' degree of freedom weakly coupled to it. Such a value, called a 'weak value', typically falls outside the eigenvalue spectrum of the operator. The appearance of unusual weak values has been associated with a unique interference structure called 'superoscillations' (band-limited functions which on a finite interval, approximate functions with spectra well outside their band). It is proposed that superoscillations play an important role in the interferences which give rise to superluminal effects. To demonstrate that, we consider a certain simple tunnelling barrier which allows a wave packet to travel in zero time and negligible distortion, a distance arbitrarily longer than the width of the wave packet. The peak is shown to result from a superoscillatory superposition at the tail. Similar reasoning applies to the dwell time. For this system, both the Wigner time (related to the group velocity) and a clock time correspond to superluminal velocities.

1. Introduction

The statistics of tunnelling particles, and more generally of scattered ones, is an instance of quantum conditional probability. For a given initial state (or class of states), we would like to calculate the *a posteriori* statistics of various observables for a given postselection (i.e. in the far future the particle is found (measured to be) on the far side of the barrier). Note that without postselection, the final state of a particle impacting a high barrier would be a superposition of a reflected and a (small) transmitted wave, and the statistics of a generic observable would depend on both.

Consider a simple 'clock' degree of freedom coupled to our particle when it is under the barrier. In the weak coupling limit (weak measurement), the expectation value of the clock degree of freedom has a simple dependence on the initial and final states. This is an instance of a 'weak value'. In section 2, some useful properties of weak values are reviewed, with some examples where they occur, and their physical significance and the related subject of superoscillations are discussed. In section 3, the conditional dwell time is shown to be the weak value of a simple clock.

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In section 4, a simple system is introduced where superluminal tunnelling is shown to occur with respect to the group-delay time as well as the conditional dwell time. In section 5, both these results are shown to result from the same superoscillatory interference.

The superluminality of the title refers to the superluminal dwell time and group delay times. The signal time, defined as the time it takes the front of an abrupt signal to arrive, is never superluminal, as required by causality. However, it does not correspond to the times measured experimentally. All the operational definitions of tunnelling times proposed to date can result in superluminal times. For extensive reviews as well as theoretical background and bibliography see [1–3].

The superluminal predictions obtained in some calculations are simply artefacts of the unphysical superluminal diffusion of the nonrelativistic Schrödinger equation. A well-known example is Hartman's explanation of the effect named after him [4], where the faster components of a (Gaussian) wave packet, by virtue of their higher energies, are transmitted preferentially through a square barrier, and since the analysis was done with the non-relativistic equation, this could lead to superluminal traversal. However, superluminality in tunnelling is generic and appears in calculations done with relativistic wave equations and with massless particles as well. It has been shown in many cases to result from a reshaping of the tail of the wave function to resemble it's central peak [5–11]. This phenomenon has been termed 'mode reshaping'. In particular, this explanation makes it clear that the information about the peak was available in the tail before the tunnelling ever occurred (i.e. given the tail of the wavefunction, one could theoretically reconstruct the rest). This effect is seen very clearly in the model system we consider here, and the effect is very dramatic—the reconstruction can be done in a region many standard deviations away from the centroid of the packet.

2. Weak values and superoscillations in the context of the two-state formalism

The two-state formalism is a time-symmetric formulation of Quantum measurement, dealing with weak measurements of pre- and post-selected systems [12–15]. It is useful in analysing experimental situations involving postselection.

In this formalism, one deals with systems prepared at a given initial time in an initial state $|i\rangle$, which are made to interact weakly with a measuring device (to be described shortly), and then at a given final time found to be in a specified final state $|f\rangle$. In practice, this usually means one performs on an ensemble, a measurement of an appropriate observable at the initial time, selects only the systems in state $|i\rangle$ (preselection), performs the weak measurement, and at the final time measures the system again to test if it is in the final state $|f\rangle$ (postselection).

A conventional ('strong') measurement can be modelled as follows. The system is made to interact with a 'pointer' degree of freedom, Q, via an interaction Hamiltonian of the form

$$H_{\text{int}} = g(t)PA, \tag{1}$$

where P is the variable conjugate to Q. The function $g(t)$ satisfies $\int g(t)\,\mathrm{d}t = 1$, and is nonzero only in a small interval. If this interval is short enough, one can assume that during this time, the free Hamiltonian can be neglected. The initial state of the pointer variable is described by the wavefunction:

$$\Phi_i(Q) = (\Delta^2 \pi)^{-1/4} \, \mathrm{e}^{-Q^2/2\Delta^2}, \tag{2}$$

and the initial state of the system is

$$|\Psi_i\rangle = \sum_k \alpha_k |a_k\rangle, \tag{3}$$

expressed in terms of the eigenfunctions of A. After the interaction the state of the system and measuring device is

$$(\Delta^2 \pi)^{-1/4} \sum_k \alpha_k |a_k\rangle \, \mathrm{e}^{-(Q-a_k)^2/2\Delta^2}. \tag{4}$$

If the width of the initial position of the pointer, Δ, is much smaller than the separation between any two eigenvalues a_k, then two different eigenstates of the system at the initial time lead to almost orthogonal states of the pointer. After amplification, the pointer is made to collapse. It is then found very close to a position corresponding to an eigenvalue of A.

The weak measurements can be described by the same model with the single change that the initial spread of the pointer position, Δ, is very large, and therefore that of P, small. Since the spread is large, the inaccuracy of single measurements has to be compensated for by large statistics. Furthermore, one also has to compensate for the reduction in the size of the ensemble due to postselection.

For postselection of state $|f\rangle$ of the system, the pointer wavefunction at the final time is the state described by equation (4), projected onto $|f\rangle$:

$$\Phi_f(Q) = (\Delta^2 \pi)^{-1/4} \sum_k \alpha_k \langle f | a_k \rangle \, \mathrm{e}^{-(Q-a_k)^2/2\Delta^2}. \tag{5}$$

Using the fact that[1]

$$\langle \mathrm{e}^{-(x-\mu_k)^2/2\sigma^2} \rangle \equiv \frac{\sum c_k \, \mathrm{e}^{-(x-\mu_k)^2/2\sigma^2}}{\sum c_k} \approx \mathrm{e}^{-(x-\langle\mu\rangle)^2/2\sigma^2}, \; \langle \mu \rangle \equiv \frac{\sum c_k \mu_k}{\sum c_k} \tag{6}$$

for all x s.t. : $|\langle\mu\rangle| \ll |x - \langle\mu\rangle| \ll \sigma$, we get:

$$\Phi_f(Q) \approx (\Delta^2 \pi)^{-1/4} \, \mathrm{e}^{-(Q-A_W)^2/2\Delta^2}, \; A_W \equiv \frac{\langle \Psi_f | A | \Psi_i \rangle}{\langle \Psi_f | \Psi_i \rangle}$$

$$\forall Q \text{ s.t.}: |A_W| \ll |Q - A_W| \ll \Delta. \tag{7}$$

The pointer may be said to show the value A_w, which is called the weak value of A with respect to the initial and final states of the system (and is independent of the initial state of the pointer). Measurement of Q yields the real part of A_w, but the imaginary part also has a physical meaning and can be measured instead (by measuring P). Note that the derivation above shows that the criterion for a weak measurement is $\Delta \gg |A_W|$ (due to our choice of units, Q and A have the same dimensions).

Besides the fact that weak values are in general strictly complex, their magnitudes can also lie outside of the range of eigenvalues. For example, it is straightforward to check, that for a spin 1/2 system preselected to be in an

[1] We use the notation $\langle \; \rangle$ for convenience, but note that this does not imply a proper average—the c_n can be any complex numbers, and $\langle\mu\rangle$ can lie outside the range of the μ_k.

eigenstate of s_x and postselected to be in an eigenstate of s_y, the weak-value of s_n, $\hat{n} = (1/\sqrt{2})(\hat{x} + \hat{y})$ is $\sqrt{2}$. In general, the deviations from the 'allowed' range of values can be arbitrarily large. In fact, for a nontrivial operator the set of weak values comprises the entire complex plane[2].

We see from the condition below equation (6), that for $\sigma \gg |\langle \mu \rangle|$, the Fourier transform of both sides of that equation should be approximately the same. The Fourier transformed equation has the form:

$$\left(\sum_n c_n \, \mathrm{e}^{\mathrm{i}\mu_n k} - c \, \mathrm{e}^{\mathrm{i}\langle \mu \rangle k} \right) \exp\left(-\sigma^2 k^2/2\right) \approx 0. \tag{9}$$

In other words,

$$\sum_n c_n \, \mathrm{e}^{\mathrm{i}\mu_n k} \approx c \, \mathrm{e}^{\mathrm{i}\langle \mu \rangle k} \text{ for } |k| < \frac{1}{\sigma}. \tag{10}$$

As noted previously, $\langle \mu \rangle$ can lie outside the range of μ_n, and in that case we see that in a small interval, a function may oscillate with a frequency which lies outside its spectrum. In fact, a function may be approximated in a small interval arbitrarily well by band limited functions (more precisely, the set of restrictions to a given interval, I, of $L_2(\mathbf{R})$ functions with Fourier spectrum in some finite interval I', is dense in $L_2(I)$). This fact was discovered by Aharonov, and was termed super-oscillations [20, 21, 23]. Other integral transforms can be applied to equation (6), and yield similar results.

Note that in the special case $|i\rangle = |f\rangle$, the weak value coincides with the usual expectation value. In this formalism it may be interpreted as the result of averaging over a complete orthonormal set of final states, but of course for this special case, the weakness condition is not really needed.

3. The dwell time as a weak value

We would like to calculate the expectation value of the time measured by a 'clock' consisting of an auxiliary system which interacts weakly with our particle as long as it stays in a given region. Furthermore, we would like to restrict the calculation only to the subensemble of particles which ultimately end up on the right of the barrier. The simplest interaction is perhaps the one defined by the Hamiltonian:

$$H_{\text{int}} = P_\tau X_{(0,L)}$$

where τ is the degree of freedom of the clock and P_τ is its conjugate momentum, and

[2] To see this, let us develop the initial and final states of the particle in terms of the eigenfunctions of the operator to be measured, A:

$$|i\rangle = \sum_k \alpha_k |a_k\rangle, |f\rangle = \sum_k \beta_k |a_k\rangle, \; (A|a_k\rangle = a_k|a_k\rangle) \tag{8}$$

Then, $A_W = \langle f|A|i\rangle / \langle f|i\rangle = \sum \beta_k^* \alpha_k a_k / \sum \beta_k^* \alpha_k$. Suppose A is nontrivial, i.e. it has more than one eigenvalue. Assume $k = 1, 2$ correspond to two of these, and take $\beta_1 = \beta_2 = (1/\sqrt{2})$. Then the two equations: $A_W = (\alpha_1 a_1 + \alpha_2 a_2)/(\alpha_1 + \alpha_2) = z$, and $|\alpha_1|^2 + |\alpha_2|^2 = 1$, are three real equations in four unknowns. They can be solved for any value of z, as can be verified easily.

$$X_{(0,L)}(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{otherwise} \end{cases}.$$

This is the effective form, for example of the potential , seen by a particle in an S_z eigenstate, in the Stern–Gerlach experiment (τ being the z coordinate, and $(0, L)$ the region of the magnetic field).

In order to use the results of the previous section to calculate the conditional dwell time, we need to generalize them in two respects:

1. The postselection is no longer (explicitly at least) of a particular final state, but of the *subspace* consisting of those states that at large positive times tend to be supported on the right side—the transmitted 'out states'.
2. In deriving the formula for the weak value of a variable we had assumed that the measurement process is short enough that the evolution of the pre- and post-selected states during its duration can be neglected. This is certainly not the case here!

The formula for the weak value of an operator, O, with respect to the postselection of the subspace spanned by the (orthonormal) set of states $\{|f\rangle\}$ is easily found to be:

$$\langle O \rangle^W_{i,\{f\}} = \frac{\langle i|P_{\{f\}} O|i\rangle}{\langle i|P_{\{f\}}|i\rangle},$$

where $P_{\{f\}} = \sum_f |f\rangle\langle f|$ is the projection operator onto the subspace spanned by $\{|f\rangle\}$.

In the case of tunnelling, and more generally, scattering, this can be simplified considerable if we use the fact that an 'in state' (a state asymptotically supported to the left of the barrier and moving to the right at large negative times, for example) becomes for large positive times an 'out state' which in turn is the sum of a reflected and a transmitted component:

$$|\Psi(t << 0)\rangle = |\Psi_{incident}\rangle, \ |\Psi(t >> 0)\rangle = |\Psi_{reflected}\rangle + |\Psi_{transmitted}\rangle.$$

If we now denote by P_T the projection operator onto the space of 'transmitted' states (i.e. 'out states' traveling to the right), and apply the last result, we find for the weak value of O:

$$\langle O \rangle^W_{i=inc, f=trans} = \frac{\langle \Psi_{inc}|P_T O|\Psi_{inc}\rangle}{\langle \Psi_{inc}|P_T|\Psi_{inc}\rangle} = \frac{\langle \Psi_{trans}|O|\Psi_{inc}\rangle}{\langle \Psi_{trans}|\Psi_{inc}\rangle}.$$

The second generalization is even simpler—we simply divide the measurement into many short ones:

$$O \simeq \sum_{j=-\infty}^{\infty} O_j, \ O_j(t) \equiv O X_{[j\Delta t, (j+1)\Delta t]}(t).$$

Assuming that during a time interval of duration Δt the evolution of the states can be ignored, we have

$$\langle O_j \rangle^W_{inc, trans} \simeq C \Delta t \frac{\langle \Psi_{trans}(j\Delta t)|O|\Psi_{inc}(j\Delta t)\rangle}{\langle \Psi_{trans}(j\Delta t)|\Psi_{inc}(j\Delta t)\rangle}.$$

The constant C is an arbitrary calibration constant of the measuring 'pointer', and we naturally set $C = 1/\Delta t$. Using the fact that $\langle \Psi_{trans}(t)|\Psi_{inc}(t)\rangle$ is independent of t, we have in the limit $\Delta t \to 0$:

$$\langle O\rangle_W = \int_{-\infty}^{\infty} \frac{\langle \Psi_{trans}(t)|O|\Psi_{inc}(t)\rangle}{\langle \Psi_{trans}(t)|\Psi_{inc}(t)\rangle} \, \mathrm{d}t = \frac{\int_{-\infty}^{\infty} \langle \Psi_{trans}(t)|O|\Psi_{inc}(t)\rangle \, \mathrm{d}t}{\langle \Psi_{trans}(0)|\Psi_{inc}(0)\rangle}.$$

For our clock, $O = X_{[0,L]}$, this formula reads:

$$E(\tau, t \to \infty|i, f) = \langle X_{[0,L]}\rangle_W = \frac{\int_{-\infty}^{\infty} dt \int_0^L \mathrm{d}x \Psi_f^*(x,t)\Psi_i(x,t)}{\int_{\infty}^{\infty} \mathrm{d}x \Psi_f^*(x,0)\Psi_i(x,0)}. \tag{11}$$

This result can also be obtained directly by a first-order perturbation theory calculation, as was shown in a previous paper [16].

Steinberg [17, 18] has arrived at this formula under similar assumptions by a somewhat different line of reasoning. He introduced the term conditional (quantum) probability for the probability distribution of a system following post-selection, and we use the notation of conditional expectation in the formula above, in the same spirit. The connection to weak values was also noted by Steinberg.

The formula is valid when τ and p_τ do not appear in additional terms in the full Hamiltonian, but the generalization is straightforward.

4. Model system: particle tunnelling through an n-delta-function potential

Olkhovsky *et al.* [19] showed that a Schrödinger particle tunnelling through a double rectangular barrier traversed the distance between the bumps instantaneously in the limit that its kinetic energy was much smaller than the height of the barrier. Unlike previous examples of superluminal tunnelling, the length of the region of superluminality consists of an arbitrarily long portion with zero potential, between the bumps. Replacing the rectangular barriers in the example discussed in [19] by delta-function potentials, the calculations can be made somewhat simpler, and are easily generalized to n arbitrary delta bumps (still using the approximation of low kinetic energy).

In this section we make a direct calculation of the transmission coefficient for the stationary scattering of a scalar particle obeying the Schrödinger equation, off a multiple delta-function potential. The Schrödinger equation is of course non-relativistic and displays an unphysical superluminal dispersion. However, the time-independent equation is the same as for the scalar relativistic wave equation, and we focus on the Schrödinger equation merely for a simple concrete interpretation. The origin of the superluminality in this case is a reshaping of the tail of the wavefunction. For a further discussion of the justification of using the Schrödinger equation for investigating superluminal tunnelling times see the review by Chiao and Steinberg [3].

4.1. *Transmission through a multiple delta-function potential*

Consider the Schrödinger equation with the following potential:

$$V(x) = \Sigma \alpha_i \delta(x - L_i); \; L_0 = 0. \tag{12}$$

The energy eigenfunctions have the form (for $x < 0$ and $x > L_n$):

$$\psi(x) = \begin{cases} A\,\mathrm{e}^{\mathrm{i}kx} + B\,\mathrm{e}^{-\mathrm{i}kx} & x < 0 \\ C\,\mathrm{e}^{\mathrm{i}kx} + D\,\mathrm{e}^{-\mathrm{i}kx} & x > L_n \end{cases} \tag{13}$$

The coefficients satisfy:

$$\begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} C \\ D \end{pmatrix}, \tag{14}$$

$$M = \prod_{i=1}^{n} \left[\beta_i \begin{pmatrix} 1 & \mathrm{e}^{-2\mathrm{i}kL_i} \\ -\mathrm{e}^{2\mathrm{i}kL_i} & -1 \end{pmatrix} + I \right], \; \beta_j = \frac{m\alpha_j}{ik} \tag{15}$$

In the limit of small kinetic energy ($|\beta_i| \gg 1$), we can drop the I matrices, as long as $n < \beta_i$. It is then straightforward to prove by induction on n that:

$$M = \prod_{1}^{n} \beta_i \begin{pmatrix} \prod_{i=2}^{n}(1 - z_i) & \prod_{i=2}^{n}(z_i^{-1} - 1) \\ -\prod_{i=2}^{n}(1 - z_i) & -\prod_{i=2}^{n}(z_i^{-1} - 1) \end{pmatrix} + O(1) \tag{16}$$

where $z_1 = 1, z_i = \mathrm{e}^{2\mathrm{i}k(L_i - L_{i-1})}$ $(i = 2 \cdots n)$

As usual, we examine the case of 'stationary scattering'. To get the (amplitude) transmission coefficient for probability current flowing from the left, t, we put $A = 1, B = r, C = t, D = 0$ in equation (13):

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + r\,\mathrm{e}^{-\mathrm{i}kx} & x < 0 \\ t\,\mathrm{e}^{\mathrm{i}kx} & x > L_n \end{cases} \tag{17}$$

and we see that $t = M_{11}^{-1}$, so:

$$\begin{aligned} t = M_{11}^{-1} &\approx \frac{\beta_1^{-1} \cdots \beta_n^{-1}}{\prod_{i=2}^{n} (1 - z_i)} = \beta_1^{-1} \cdots \beta_n^{-1} \frac{\prod_{i=2}^{n} z_i^{-\frac{1}{2}}}{\prod_{i=2}^{n} (z_i^{-\frac{1}{2}} - z_i^{\frac{1}{2}})} \\ &= \frac{\prod \beta_i^{-1}}{(-2i)^{n-1}} \frac{\mathrm{e}^{-\mathrm{i}kL_n}}{\prod_{i=2}^{n} \sin\left(k(L_i - L_{i-1})\right)} \end{aligned} \tag{18}$$

The stationary phase formula for the delay time, τ_g:

$$\tau_g \equiv \hbar \frac{\partial}{\partial E} \arg(t), \tag{19}$$

yields the value $\tau_g = -(mL/\hbar k) = -(L/v(k))$ for the delay, which cancels the time for a free particle, and we get an overall zero time for tunnelling. Since this is true for all k, it should be true for an arbitrary wave packet, *as long as the stationary phase approximation holds*. The condition for that is derived in the next subsection.

4.2. *The condition for superluminal tunnelling of a packet*

Restated for wave packets, our results so far can be summarized as:

$$\Psi(x, t) = \begin{cases} \int g(k)\left(\mathrm{e}^{\mathrm{i}kx} + r(k)\,\mathrm{e}^{-\mathrm{i}kx}\right)\mathrm{e}^{-\mathrm{i}\omega(k)t}\,\mathrm{d}k & x < 0 \\ \int(-\mathrm{i}k)C(k)g(k)\,\mathrm{e}^{\mathrm{i}k(x-L_n)}\,\mathrm{e}^{-\mathrm{i}\omega(k)t}\,\mathrm{d}k & x > L_n \end{cases}, \tag{20}$$

and

$$C(k) = \frac{\prod \beta_i^{-1}}{-\mathrm{i}k(-2i)^{n-1} \prod_{i=2}^{n} \sin\left(k(L_i - L_{i-1})\right)}. \tag{21}$$

When Δk is sufficiently small, the diffusion can be ignored and $C(k)$ can be considered constant (as will be shown shortly). Then we can again separate out the time dependence of the wavefunction and the spatial part can be written:

$$\psi(x) = \begin{cases} \phi(x) & x < 0 \\ C\phi'(x - L_n) & x > L_n \end{cases} \tag{22}$$

where $\phi(x)$ in the two regions is related through analytic continuation.

If $\phi(x) = R(x)\,\mathrm{e}^{\mathrm{i}S(x)}$ where $R(x)$ is large and slowly varying in the region $|x - x_0| < \Delta x$ and $S(x)$ goes through a few cycles there, then the time of arrival distribution of the transmitted packet will be approximately that of the incoming one, shifted by $-L/\langle v\rangle$. Note also that this is also true for a mixed state which can be decomposed into various pure states with this property.

Let us now find the explicit condition for $C(k)$ to be approximately constant, for the case where $L_j = (j/n)L$, $\alpha_j = \alpha$, and as before, $n < |\beta| = m\alpha/k$. In this case, we have

$$C(k) = \frac{\left(\dfrac{\mathrm{i}k}{m\alpha}\right)^n}{-2\mathrm{i}k(2\mathrm{i}\sin kL/n)^{n-1}} \tag{23}$$

Using the fact that $(x/\sin x) = 1 + (x^2/6) + O(x^4)$, we get:

$$C(k) = -\frac{1}{m\alpha}\left(\frac{n-1}{2Lm\alpha}\right)^{n-1}\left(1 + \frac{1}{6}\left(\frac{kL}{\sqrt{n-1}}\right)^2 + O\left(\left(\frac{kl}{n-1}\right)^3\right)\right) \tag{24}$$

Thus, $C(k)$ will be approximately constant if the spectrum of the wave packet is limited to k such that $|k| \approx \sqrt{n-1}/L$. In other words, $\Delta k \approx \sqrt{n-1}/L$, or $\Delta x \approx (L/\sqrt{n-1}$. This means that the length of the barrier can be arbitrarily longer than the 'length' of the tunnelling packet as usually defined (standard deviation of the x coordinate), the penalty paid being an exponential suppression of the amplitude.

4.3. *Calculation of the dwell time*

It is interesting to compare the 'group delay' (which is zero in the low k limit) with the dwell time. For the sake of simplicity, we shall deal with the case $n = 2$. A direct calculation of the dwell time of the transmitted component can be made by calculating the transmission coefficient after adding a potential which is constant over the region between the delta spikes, and vanishing outside it. We get:

$$t \simeq \beta^{-2}\frac{\mathrm{e}^{-\mathrm{i}kL}}{-2\mathrm{i}\sin k'L},$$

where $k' = \sqrt{2m(E - V_0)/\hbar}$ and V_0 is the value of the potential between the deltas. Clearly, $(\partial \arg(t)/\partial V_0) = 0$, and the (conditional) dwell time is zero as expected.

A direct calculation of the dwell time using formula (11) also shows that it tends to zero in the low energy limit.

5. Superluminality and its relation to interference in the tail of the wavefunction

The calculation of the transmission coefficient, t, can also be done in a way more suggestive of superoscillations. Let us explain this for the case of two delta functions (the $n = 2$ case in equation (12), 'Fabry Perot interferometer').

Suppose a quasimonochromatic wave packet with wave number k arrives at the first delta spike. The transmitted component is the same as the original wave, except for an attenuation and phase which are independent of k. At the second delta spike, the wave splits into a (approximately unattenuated) reflected wave and a transmitted one which is apart from a k-independent multiplicative constant the same as the impinging wave. The reflected component is again reflected at the first delta, and arrives at the second delta with an additional phase of $2kL$, but with approximately the same amplitude as the original transmitted wave. In a like manner, one gets additional transmitted waves with additional phases of $2nkL, n = 2, 3, \ldots,$ and amplitudes which decrease very slowly. Thus we get the following formal sum for the resulting amplitude of the wave (up to a multiplicative constant):

$$\sum_n \mathrm{e}^{\mathrm{i}kx}(\mathrm{e}^{\mathrm{i}k2L})^n = \frac{\mathrm{e}^{\mathrm{i}kx}}{1-\mathrm{e}^{2\mathrm{i}kL}} = \mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{e}^{-\mathrm{i}kL}}{-2\mathrm{i}\sin kL} \tag{25}$$

which is in agreement with our previous calculation. This is an example of superoscillations since a sum of positive wavevectors results in a negative one (or, equivalently, a sum of positive shifts which results in a negative shift)[3]. This is true in the following sense: for $|k| \ll 1/L$ the denominator of the right-hand side can be considered constant. However, in such a small interval the function does not really oscillate, so it really does not have a well-defined frequency. To really speak about superoscillations we need to have a large number of delta-functions. The sum for the case $n > 2$ factors into $n - 1$ sums of the above form, in the low kinetic energy limit, since to leading order in β^{-1}, the only contributions are from waves which tunnel through each delta only once, but may be reflected any number of times between consecutive deltas. We then reproduce the results of subsection 4.2, where we had the weaker condition $k < \sqrt{n-1}/L$, allowing the function to complete many oscillations in the region.

The same kind of calculation can be done for the wavefunction of our clock's 'pointer'. Then this superoscillatory sum would correspond to a series of positive

[3] The sum in equation (25) actually diverges, the physical reason being that we have neglected the attenuation of the amplitude, in order to maintain consistency with the low kinetic energy approximation we have used so far. For the case $n = 2$ it is easy to evaluate equation (15) without resort to that approximation, and the resulting transmission amplitude is:

$$t(k) = \frac{\beta^{-2}}{\left(1+\dfrac{2}{\beta}+\dfrac{1}{\beta^2}\right)e^{2\mathrm{i}kL}} \tag{26}$$

Similarly, the sum on the left of equation (26) should be replaced by the exact one:

$$(1+\beta)^{-2}\sum_{j=0}^{\infty}\left[\left(\frac{\beta}{1+\beta}\right)^2 \mathrm{e}^{2\mathrm{i}kL}\right]^j = \frac{\beta^{-2}}{\left(1+\dfrac{2}{\beta}+\dfrac{1}{\beta^2}\right)-\mathrm{e}^{2\mathrm{i}kL}} \tag{27}$$

time readings interfering to give zero. This derivation shows that for this system, the dwell and group-delay times not only coincide, but are also described by the same mechanism. Note that since the clock is described by a projection operator, with eigenvalues 0 and 1, the superoscillatory sum cannot describe the series of equation (6). However, it can be interpreted as the product of such series corresponding to the decomposition of the clock projector into a sum of projectors onto the same spatial interval multiplied by appropriate time intervals (the weak value of a sum is the sum of the weak values).

The group delay in tunnelling through a thick barrier follows from the fact that under the barrier, no phase accumulates, and the entire phase shift comes from the boundaries and is practically independent of the thickness. For cases where interference with a delayed wave takes place, a few authors [5–11] have suggested a different mechanism. In Chiao and Steinberg's words [3]: 'If destructive interference is set up between part of the wave traveling unimpeded and part which has suffered a delay Δt due to multiple reflections, one has $\Psi_{out}(t) = \Psi_{in}(t) - \xi\Psi(t - \Delta t) \approx (1 - \xi)\Psi_{in}(t) + \xi\Delta t\, \mathrm{d}\Psi_{in}(t)/\mathrm{d}\, t \approx (1 - \xi)\Psi_{in}(t + \xi\Delta t/(1 - \xi))$, which is already a linear extrapolation into the future. In cases where the dispersion is sufficiently flat, as in a bandgap medium, the extrapolation is in fact surprisingly better than this first-order approximation. As was suggested by Steinberg [7] and recently discussed more rigorously by Lee and Lee [9] and Lee [11], this implies that even a simple Fabry–Perot interferometer exhibits superluminality when excited off resonance' [presumably, $\xi \ll (1/\Delta t\Psi'(t))$]. Another physical interpretation of the mode reshaping process is also suggested in [8]. We would like to explain this 'better than first-order' approximation. Let us instead look at the momentum wavefunction. A spatial shift corresponds to a linear shift in this function. A positive spatial delay would correspond to a linear shift steeper than one, and the converse for a negative delay. In the Taylor expansion of the transmission coefficient for the momentum wave function, the zero term is insignificant, the second corresponds, as just explained, to the spatial shift, and the higher give the distortion. When many waves with *large* and evenly distributed shifts interfere, their sum is for a wide range of momenta, zero, and in particular momentum independent. In other words, the momentum wavefunction is flat for a wide band of frequencies. This corresponds to a much better than first-order approximation of the spatial wavefunction, as can be seen in the special case of the system described in this paper.

6. Discussion

It is important to note that when one speaks of the time it takes an atom to emit a photon, one usually means the standard deviation of the time of emission. This value does not prescribe any bound on the length of time over which the wavefunction of the photon traversing a point is analytic. One might imagine the emitting atom sending an arbitrarily long tail ahead of the actual peak of a photon. An appropriate tunnelling device could reshape this tail to resemble the peak, and the photon could be detected there well in advance of the time the peak should have reached it.

The probability of finding the photon there so early is not greater than in the absence of the barrier, as the reshaped tail still has the same probability as the unreshaped one—with probability close to 1, the photon will be reflected back, and never reach this point. However, if one considers a large number of emitting atoms, and such a barrier, then for the subensemble of photons that manage to traverse the barrier, the distribution of the measured arrival times should be approximately the same as in the absence of the barrier, but shifted by the time it takes to traverse its length.

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