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Interplay of Aharonov–Bohm and Berry phases

B. Reznik and Y. Aharonov [1]

School of Physics and Astronomy, Beverly and Raymond Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel

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An interplay of the Aharonov–Bohm phase and the Berry phase appears as a fluxon circulates an extended (quantum) charge distribution. For a fluxon–charge system in a superconductor, we will show how the interplay of the phases leads to a net topological effect. The realization of the effect in the Higgs model is discussed.

1. Introduction

Topological phases can arise as a system undergoes a cyclic motion. The phase accumulated by a charge encircling a magnetic fluxon is the Aharonov–Bohm (AB) effect [1] (AB), or, for a magnetic moment encircling a line of charge, the Aharonov–Casher (AC) effect [2]. Cyclic motion of a state in Hilbert space can also give rise to the Berry phase [3]. The latter phase may be distinguished from the AB and the AC phases. While the AB (or AC) phase appears in the Hamiltonian of any physical system containing a fluxon and a point charge (or a magnetic moment and a line of charge), the Berry phase applies only in the adiabatic limit, i.e. if the system includes "heavy" degrees of freedom whose motion can be regarded as adiabatic.

Our purpose is to draw attention to an interplay of the AB (or AC) phase and the Berry phase. It occurs in a charge–fluxon system when the charge (or fluxon) is in the ground state of some potential and its location is therefore smeared. The interplay of the phases is manifested as the heavy fluxon goes through the "cloud" of charge. As will be shown below, in this case, the motion of the fluxon induces adiabatic modifications in the ground state which leads to a Berry phase complementing the AC phase collected by the fluxon. (A similar interplay of the AB and Berry phases occurs if the fluxon is smeared and the charge is heavy.) We will demonstrate that this interplay appears even if the charge is screened and the fluxon is embedded in a superconductor. A non-trivial phase, collected as the fluxon encircles the charge, emerges as a sum of the usual AC part and an additional Berry phase. The two phases complement each other, yielding an overall topological phase identical to the "free" phase accumulated when there is no screening. We study the interplay of the phases in a superconductor model, and examine its implications for the interaction of a vortex line and an external charge in the Higgs model. Other topological aspects of the problem are discussed elsewhere [4].

2. Interplay of topological phases

The interplay of AB and Berry phases takes its simplest form in the following system. Consider a system of two non-interacting particles of masses m, M with coordinates and momenta $\boldsymbol{r}$, $\boldsymbol{p}$ and $\boldsymbol{R}$, $\boldsymbol{P}$ respectively in two dimensions. The Hamiltonian and the wave function of the system are

$$H=\frac{\boldsymbol{p}^2}{2m}+V(\boldsymbol{r})+\frac{\boldsymbol{P}^2}{2M} \qquad (2.1)$$

and

[1] Also at Department of Physics and Astronomy, University of South Carolina, Columbia, SC 29208, USA.

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$$\Psi(\boldsymbol{r},\boldsymbol{R})=\psi_0(\boldsymbol{r})\phi(\boldsymbol{R}) \,, \tag{2.2}$$

respectively. The particle of mass m is assumed to be in the ground state $\psi_0(\boldsymbol{r})$ (or any stationary bound state) of some potential $V(\boldsymbol{r})$, while the second particle (M) is free.

Compare this two-particle system with one in which particle m has a charge q, and particle M has a corresponding (quantized) fluxon of magnitude ch/q. The new electron–fluxon system is generated from the old Hamiltonian (2.1) by the unobservable gauge transformation

$$\Psi(\boldsymbol{r},\boldsymbol{R})\rightarrow\Psi'(\boldsymbol{r},\boldsymbol{R})=\exp[\mathrm{i}\Lambda(\boldsymbol{r}-\boldsymbol{R})]\,\Psi(\boldsymbol{r},\boldsymbol{R}) \,, \tag{2.3}$$

where Λ is the angle between the vector $(\boldsymbol{r}-\boldsymbol{R})$ and the $\hat{x}$ axis [#1]

$$\Lambda(\boldsymbol{r}-\boldsymbol{R})\equiv\arctan\left(\frac{y-Y}{x-X}\right), \tag{2.4}$$

with $\boldsymbol{r}=\boldsymbol{r}(x,y)$ and $\boldsymbol{R}=\boldsymbol{R}(X,Y)$. This transformation yields the new Hamiltonian

$$H'=\frac{[\boldsymbol{p}+\hbar\nabla_r\Lambda(\boldsymbol{r}-\boldsymbol{R})]^2}{2m}+V(\boldsymbol{r})+\frac{[\boldsymbol{P}+\hbar\nabla_R\Lambda(\boldsymbol{r}-\boldsymbol{R})]^2}{2M}\,, \tag{2.5}$$

with the induced vector potentials

$$\nabla_r\Lambda(\boldsymbol{r}-\boldsymbol{R})=-\nabla_R\Lambda(\boldsymbol{r}-\boldsymbol{R})=\frac{\hat{\phi}}{|\boldsymbol{r}-\boldsymbol{R}|}\,, \tag{2.6}$$

where $\hat{\phi}$ is the unit vector orthogonal to $\boldsymbol{r}-\boldsymbol{R}$.

Note that the AB and AC vector potentials of the system above can be expressed by

$$\boldsymbol{A}_{\mathrm{AB}}=\boldsymbol{A}(\boldsymbol{r}-\boldsymbol{R})=\frac{c\hbar}{q}\nabla_r\Lambda\,,$$

$$\boldsymbol{A}_{\mathrm{AC}}=\frac{1}{q}\boldsymbol{\mu}\times\boldsymbol{E}(\boldsymbol{R}-\boldsymbol{r})=\frac{c\hbar}{q}\nabla_R\Lambda\,,$$

where $\boldsymbol{\mu}=(ch/q)\boldsymbol{e}_z$ is the magnetic moment of the fluxon, $\boldsymbol{E}(\boldsymbol{R}-\boldsymbol{r})$ is the electric field at the location of the fluxon, and $\boldsymbol{A}(\boldsymbol{r}-\boldsymbol{R})$ is the electromagnetic vector potential generated by the fluxon at the location of the charge. Therefore, the transformed Hamiltonian H' is indeed identical to the Hamiltonian of an electron–fluxon system, and the vector potentials in (2.5) admit a natural interpretation. The charge interacts with the AB vector potential $\boldsymbol{A}$ generated by the fluxon, while the fluxon interacts with the AC vector potential $\boldsymbol{\mu}\times\boldsymbol{E}$ due to the local electric field. (Of course the same Hamiltonian also describes a system in which the fluxon is bound and the charge is free.) We know that this system is topologically trivial, being equivalent to the non-interacting Hamiltonian (2.1). Nevertheless, it is instructive to see how a trivial phase emerges.

If the fluxon's trajectory is limited to the region outside the "cloud" of charge, it will clearly accumulate a phase of 2π, since the path encircles the total charge. Consider, however, a closed path that crosses the smeared charge. If the motion of the fluxon is adiabatic, with respect to the characteristic time scale of the ground state, the effect is the same as a classical charge distribution. In this case, we have a non-trivial AC phase corresponding to the fraction of the charge the path encloses. On the other hand, we know that this system cannot give rise to a non-trivial topological effect since its Hamiltonian is gauge equivalent to a system of two non-interacting particles.

This apparent contradiction is reconciled by an additional phase due to the motion in Hilbert space, i.e., by Berry phase. With respect to the ground state, the adiabatic motion of the fluxon can be viewed as a slowly varying external source. Then, as is well known, the motion in Hilbert space generates the "Berry" vector potential

$$A_{\mathrm{Berry}}=\mathrm{i}\langle\Psi_{\mathrm{g}}^{(R)}(r)\,|\,\nabla_R\Psi_{\mathrm{g}}^{(R)}(r)\rangle\,. \tag{2.7}$$

For the system above,

$$\Psi_{\mathrm{g}}^{(\mathrm{R})}(\boldsymbol{r})=\exp[\mathrm{i}\Lambda(\boldsymbol{r}-\boldsymbol{R})]\,\psi_0(\boldsymbol{r})\,,$$

and $\boldsymbol{R}$ the location of the fluxon, is a slow parameter. The Berry vector potential is in fact the negative of the AC vector potential. Therefore, the Berry phase in this case always cancels the AC phase exactly. Thus, although the AC phase (alone) could be non-trivial topologically, it can not be observed, and the total phase accumulated by this system is trivial as expected.

A similar interplay of Berry and the AB phase ap-

[#1] Although this gauge function might seem completely trivial, it is not. If $\Psi(r_1,r_2)$ describes two identical bosonic particles, the transformed wave function Ψ' has the symmetry of two fermions.

pears if a heavy free charge circulates a smeared (point-like quantized) fluxon. As the charge slowly crosses the "cloud" of flux we have a non-trivial AB phase that is cancelled by the Berry phase.

While the interplay of the phases in the system above yields a net trivial phase, in other systems it can lead to a non-trivial effect. In ref. [4] it is shown how the interplay allows a topological approach to a system of an extended charge and a fluxon carrying half a quantum of flux. In the following we discuss the generic system of a fluxon and a (non-quantized) charge in a medium that screens the field strengths, and show that the interplay leads to a net topological effect.

3. Berry and AC phases in a superconductor model

Consider an external charge Q_0 and a fluxon of magnitude (hc/q) in a medium of particles of charge q that screens the electromagnetic field. Despite the screening of the electric field a cancellation mechanism, very similar to the one described above, guarantees the usual AC phase, $2\pi Q_0/q$, when the fluxon encircles Q_0. A possible realization of such a system is a type-II superconductor containing a fluxon and some external charge Q_0 with $Q_0 \bmod(q) \neq 0$ (where q is the condensate charge). As the charge encircles the fluxon at a large distance, a non-trivial AB phase, $\Phi_{AB}=2\pi Q_0/q$, arises from the interaction with the AB vector potential. This interaction is unchanged by the superconductor. On the other hand, consider the phase accumulated as the fluxon encircles the charge at a large distance. The interaction of the fluxon via the AC vector potential $\boldsymbol{\mu}\times\boldsymbol{E}$ is screened by the superconductor. In this case, it is less obvious that the system accumulates the same phase. In a previous paper [5] [#2] we have argued that in the latter process the same non-trivial phase $\Phi_{AB}=\Phi_{AC}=2\pi Q_0/q$, is accumulated, i.e. the truly topological phase is unaffected by the screening of $\boldsymbol{E}$. This was demonstrated by constructing a simplified model for the system. As will be shown in the following, this model also demonstrates the interplay of AC and Berry phases.

It is amusing to point out that similar Berry phases must be invoked under other circumstances. Consider the effect of adding a neutral external particle of large mass M instead of an external charge Q_0. The medium will screen the effect of the mass by inducing some local charge density around it. The combined gravity+electrostatic force on the Cooper pairs will vanish, but in this case the electric field in the superconductor will be nonzero! Consequently, the fluxon will interact with this electric field. The AC phase accumulated by the fluxon will in general be non-trivial (depending on the ratio GmM/q^2, where m is the mass of a Cooper pair). The total phase accumulated by the system must be zero, however, since the unit charge of the medium is quantized properly with respect to the fluxon. Indeed, as we shall verify below, the cancellation of the AC phase is due to an additional non-trivial Berry phase. This phase results from the distortion of the superconductor's wave function under the influence of the external slowly moving fluxon. To see how this phase is generated, we shall introduce an effective model which describes the quantum behavior of the induced Cooper pairs (or holes) that screen the external field.

Let Q_0 denote the external charge which is inserted into the superconductor. An effective Hamiltonian that describes the effect of screening the electric field of this charge reads [5,6]

$$H_{sc}=\frac{1}{2\mu}(\hat{Q}+Q_0)^2-\omega\cos\theta\,, \tag{3.1}$$

where $\hat{Q}=-\mathrm{i}q\,\partial/\partial\theta$ is the momentum conjugate to θ and ω, μ are constants. The ground state is given (in the limit $\omega\mu\gg q^2$) by

$$\psi_0(\theta)=\sum_{N=-\infty}^{\infty}\exp\left(-\mathrm{i}(\theta-2\pi N)\frac{Q_0}{q}\right)\times u_0(\theta-2\pi N)\,, \tag{3.2}$$

where

$$u_0(\theta)=\left(\frac{\sqrt{\omega\mu}}{\pi q}\right)^{1/4}\exp\left(-\frac{\sqrt{\omega\mu}}{2q}\theta^2\right). \tag{3.3}$$

Defining the electric field operator as

$$\boldsymbol{E}=\frac{\hat{Q}+Q_0}{2\pi r}\boldsymbol{e}_r \tag{3.4}$$

(r, the radial distance, is a c number), it is easily verified that since $\langle\hat{Q}\rangle=-Q_0$, $\langle\boldsymbol{E}\rangle=0$, and $\langle\boldsymbol{E}^n\rangle$

[#2] A discussion from another point of view is presented in ref. [6].

(where n is some integer) does not depend on Q_0. However the modular electric field

$$\left\langle \exp\left(\mathrm{i}\frac{1}{q}\oint E\cdot \mathrm{d}l\right)\right\rangle = \exp\left(\mathrm{i}\frac{2\pi Q_0}{q}\right) \qquad (3.5)$$

does depend on the external charge Q_0 and gives an AC phase as if the electric field were not screened. Therefore, while the superconductor screens (exponentially, in the full theory) any moment of the external electric field, the modular electric field is not screened.

We now couple this system, of a charge Q_0 in a superconductor to a fluxon. We assume that the fluxon can be approximated by a point-like particle of effective mass M_f that interacts only with the induced Cooper pairs and the charge Q_0. Consider first the case $Q_0=0$. As in section 2 above the Hamiltonian must be equivalent, modulo a gauge transformation, to the interaction-free Hamiltonian

$$H=H_{\mathrm{sc}}(Q_0=0)+\frac{P_\phi^2}{2M_f}, \qquad (3.6)$$

where we have neglected the irrelevant radial motion of the fluxon. Performing the gauge transformation

$$\Psi(\theta,\phi)\to\exp\left(\mathrm{i}\frac{1}{q}\hat{Q}\phi\right)\psi_0(\theta)\chi(\phi) \qquad (3.7)$$

(analogous to (2.3)), where $\chi(\phi)$ is the fluxon's wave function and ϕ is the angular coordinate of the fluxon (conjugate to P_ϕ), we find that the Hamiltonian transforms to

$$H'=\frac{\hat{Q}^2}{2\mu}-\omega\cos(\theta+\phi) + \frac{[P_\phi+(\hbar/q)\hat{Q}/r]^2}{2M_f}. \qquad (3.8)$$

This Hamiltonian can be interpreted as in our former discussion above. The fluxon enters via interactions through the AC vector potential (the third term above), while the potential for the induced charge (second term above) is shifted by the angle ϕ. This modified potential term gives rise to a Berry phase. With no external charge, the sum of the two phases cancels, as is seen from the explicit wave function (3.7). So far we have indeed performed a pure gauge transformation. The same cancellation can be reached also by considering the effect of the gauge transformation on the Cooper pair wave function (in the n-representation). The gauge transformation (3.7) can be written

$$\Psi'=\exp\left(\mathrm{i}\frac{1}{q}\hat{Q}\phi\right)\chi(\phi)\sum_n\exp(\mathrm{i}n\theta)u_n = \chi(\phi)\sum_n\exp[\mathrm{i}n(\theta+\phi)]u_n. \qquad (3.9)$$

The coefficients u_n are merely the n-particle Schrödinger wave functions. In a state of definite number, say n, of Cooper pairs, the fluxon moves on a path that encloses the charge and accumulates an AC phase $-2\pi n$. Eq. (3.9) states that as the angle ϕ changes by 2π, the nth component of the many-particle wave function is to be shifted by a phase $+2\pi n$.

Next consider $Q_0\neq 0$. The Hamiltonian that includes the effect of the external charge Q_0 can be obtained by shifting the induced charge operator $\hat{Q}\to\hat{Q}+Q_0$. The Hamiltonian of the total system than reads

$$H_{\mathrm{tot}}=\frac{(\hat{Q}+Q_0)^2}{2\mu}-\omega\cos(\theta+\phi) + \frac{[P_\phi+(\hbar/q)(\hat{Q}+Q_0)/r]^2}{2M_f}. \qquad (3.10)$$

If we regard the motion of the fluxon as adiabatic, the Hamiltonian (3.10) indicates three contributions to the total phase of the wave function. The fluxon interacts through an AC vector potential consisting of two parts. One is the vector potential generated by the induced charge (given by $(\hbar/q)\hat{Q}/r$), and the second is the contribution of the external charge (given by $(\hbar/q)\ Q_0/r$). The fluxon also acts on the superconductor as an external source, which shows up in (3.10) by the shift of the potential term to $\cos(\theta+\phi)$. This shift is the source of the Berry phase. The solution of the Schrödinger equation reads

$$\Psi(\theta,\phi)=\exp\left(\mathrm{i}\frac{1}{q}(\hat{Q}+Q_0)\phi\right)\psi_0(\theta)\chi(\phi). \qquad (3.11)$$

Clearly, as the fluxon completes one revolution, the wave function is shifted by $\exp(2\pi\mathrm{i}Q_0/q)$. The total phase can be separated according to the discussion above into the following three parts:

$$\Phi_{\rm tot}=\Phi^{\rm ext}_{\rm AC}+\Phi^{\rm ind}_{\rm AC}+\Phi_{\rm Berry}$$
$$=\frac{2\pi Q_0}{q}-\frac{2\pi Q_0}{q}+\frac{2\pi Q_0}{q}=\frac{2\pi Q_0}{q}\,. \qquad (3.12)$$

We conclude that two equivalent interpretations of the topological effect arising from the interaction of a fluxon with an unquantized charge in a superconducting medium are possible. Either one says the local average electric field vanishes so that the total AC phase, due to the first two terms in (3.12) is zero. The topological phase, in this case, is given by a nontrivial Berry phase. Or, one can say that the effect of the Berry phase is to cancel the AC phase arising from the contribution of the induced charge ($\Phi^{\rm ind}_{\rm AC}+\Phi_{\rm Berry}=0$). What is left is the local AC phase generated by the external charge.

4. Interplay of phases in the Higgs model

Systems similar to the one discussed above have been studied in the framework of field theories with broken symmetry. The vacuum of those theories admits a topological vortex with a confined flux which is the relativistic analog of the vortex of a type-II superconductor. Since the vacuum cannot screen the modular electric field of an external unquantized charge (with respect to the charge of the condensed field), it was argued that as the vortex line circulates an external charge it must collect a nontrivial phase and give rise to an observable interference [2,7]. This effect could be used for example to detect the "hair" of a black hole [7,8]. We would like to indicate that a dynamical demonstration of the type discussed here including the analogue of Berry's phase is possible also in the case of a "relativistic superconductor". To this end we will show how the usual Abelian Higgs model reduces in the presence of an external non-dynamical charge to a generalized version of the model discussed above.

Consider the Higgs Lagrangian in 2+1 dimensions

$$\mathscr{L}=-\tfrac{1}{4}F^2+|(\partial_\mu+ieA_\mu)\phi(x)|^2+V_H(\phi)$$
$$+J_\mu A^\mu\,, \qquad (4.1)$$

where ϕ is the Higgs field, V_H is the Higgs potential and $J_\mu=(J_0,0,0)$. In our non-relativistic model we have seen that the non-trivial phase could be derived from the interaction of the fluxon with the radial fluctuations of the electric field. Therefore, we consider a spherical symmetric approximation for the fields. In the low energy limit the Higgs field is given by $\phi(r,t)=\phi_0\exp[i\alpha(r,t)]$ $(V_H(\phi_0)=0)$. The Hamiltonian in this approximation reads

$$H=\int_0^\infty 2\pi r\left(\frac{E^2}{2}+\frac{\Pi_\alpha}{4\phi_0^2}+\phi_0^2|\nabla\exp(i\alpha)|^2\right)dr\,, \qquad (4.2)$$

with $\Pi_\alpha=\partial\mathscr{L}/\partial(\partial_0\alpha)$ and $A=0$. The additional constraint (Gauss law) reduces to an equation for the radial electric field

$$E(r,t)=\frac{e_r}{2\pi r}\int_0^r 2\pi r'(e\pi_\alpha+J_0)dr'\,. \qquad (4.3)$$

Finally, we make the theory discrete by letting the parameter r take the values r_i separated by the intervals ϵ_i, such that $2\pi\epsilon_i r_i=1$ where $i=1,2,...,\infty$. The radial electric field at the point r_i is given by

$$E_i=\frac{1}{2\pi r_i}\sum_{i=1}^{\infty}(e\Pi_i+J_{0i})$$
$$=(2\pi r_i)^{-1}(Q_i+Q_0)\,, \qquad (4.4)$$

where $Q_i=\sum_i e\Pi_i$ is the total charge enclosed in a circle of radius r_i, and Q_0 is the total external charge. The Hamiltonian on the "lattice", denoted as H', becomes

$$H'=\sum_i\frac{(Q_i+Q_0)^2}{8\pi^2r_i^2}+\frac{\Pi_i^2}{4\phi_0^2}$$
$$+2\frac{\phi_0^2}{\epsilon_i^2}[1-\cos(\alpha_i-\alpha_{i-1})]\,. \qquad (4.5)$$

To quantize this model we impose the commutation relations $[\alpha_i,\Pi_j]=i\hbar\delta_{ij}$ and substitute $\Pi_i=-i\hbar\partial/\partial\alpha_i$ to obtain the Schrödinger equation of the effective model

$$i\hbar\frac{\partial}{\partial t}\Psi(\alpha_i,t)=H'\Psi(\alpha_i,t)\,. \qquad (4.6)$$

Comparing (4.5) with the effective Hamiltonian (3.1) we note that the resulting Hamiltonian is (up to the term Π_i^2/ϕ_0^2) merely a sum of effective Hamiltonians of the same form as (3.1). The charge $\hat{Q}$ is replaced by $\hat{Q}_i$, the total charge up to r_i and $\theta_i=\alpha_{i+1}-\alpha_i$ is a local version of the angular variable θ conjugate to $\hat{Q}$. Indeed, since $[\theta_i,\hat{Q}_j]=ie\hbar\delta_{ij}$, we could have regarded $\hat{Q}_i$ and θ_i as the basic conjugate variables. For $r_i < e\phi_0$ we can neglect the term Π_i^2/ϕ_0^2. In this region we can expect the electric field to behave as in the model above. For $r_i \gg e\phi_0$ this term dominates, since at a large distance from the external charge local fluctuations contribute more than fluctuations associated with screening. However, the basic features of the simple model remain. Since the variables θ_i are periodic, the charge $\hat{Q}_i$ enclosed in a shell of radius r_i is quantized with respect to the basic unit of charge $q = e\hbar$ and so

$$\left\langle \exp\left(\mathrm{i}\frac{1}{e\hbar}\oint E\cdot \mathrm{d}l\right)\right\rangle = \left\langle \exp\left(2\pi\mathrm{i}\,\frac{Q_i+Q_0}{e\hbar}\right)\right\rangle = \exp\left(2\pi\mathrm{i}\,\frac{Q_0}{e\hbar}\right). \tag{4.7}$$

To derive the modifications due to the presence of a fluxon we repeat our argument in section 3. The (pure) gauge transformation $U = \exp\{\mathrm{i}[(1/e\hbar)Q_i\phi]\}$ introduces an interaction with a fluxon at the point (r_i, ϕ). This transformation affects only the potential term containing θ_i, and shifts it to $\cos(\theta_i + \phi)$. We therefore see that while the motion of the fluxon induces only a local change in the Hamiltonian, the modification of the potential due to the motion of the fluxon will give rise to a Berry phase. Alternatively, one could say that the Berry phase cancels the part of the AC phase due to the Cooper pair screening the external charge. The net phase, due to the interaction of the vortex with the electric field induced by the external charge, remains.

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