Quantum Limitations on Superluminal Propagation

Yakir Aharonov,[1] Benni Reznik,[2] and Ady Stern[3]
[1]*School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel*
and Department of Physics, University of South Carolina, Columbia, South Carolina 29208
[2]*Theoretical Division, T-6, MS B288, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
[3]*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
(Received 7 April 1998)

Unstable systems such as media with inverted atomic population have been shown to allow the propagation of analytic wave packets with group velocity faster than that of light, without violating causality. We illuminate the important role played by unstable modes in this propagation and show that the quantum fluctuations of these modes and their unitary time evolution impose severe restrictions on the observation of superluminal phenomena. [S0031-9007(98)07063-X]

PACS numbers: 03.65.Sq, 42.50.–p, 42.50.Lc

A famous consequence of Einstein's special theory of relativity is the principle of causality: signals cannot travel faster than light. Nevertheless, it has been known for some time that under certain conditions the group velocity of an electromagnetic wave packet can be arbitrarily large and its energy positive, but yet no conflict arises with causality: the signal velocity always remains smaller than the velocity of light in vacuum. At least two classes of such models allowing "causal" superluminal behavior are known: the first is closely related to tunneling phenomena; the second arises for unstable systems of atoms under an inverted population condition [1].

While in the first case the "superluminal" transmission of waves or particles through a barrier is exponentially suppressed, in the second case, within a semiclassical calculation, wave packets travel with a superluminal group velocity for unlimited distance, with negligible attenuation and dispersion. The second case results from instabilities in the initial state of the radiating system, for example, a scalar field initially in a "false vacuum" state [2]. Recently, an optical experiment studying superluminal group velocities for waves in a medium with inverted population was suggested [3,4]. While the discussion was semiclassical, it was further suggested [4] that with a full fledged quantum mechanical treatment one would possibly obtain stable tachyoniclike quasiparticle excitations in the inverted medium.

In this Letter we examine classical and quantum aspects of superluminal group velocities in unstable systems. Classically, a peak of an analytic wave packet may travel from point A to point B faster than the speed of light, since its shape at B can be fully reconstructed from the part of the wave packet which is causally connected to point B. We show that quantum mechanically, analyticity of the wave packet is not the only condition needed for such a reconstruction. Rather, the part of the wave packet which is causally connected to point B must contain many photons or large enough energy in the unstable modes. This condition strongly suppresses superluminal effects in the limit that the system contains only a few photons. The exponential suppression of superluminal effects characterizes both the tunneling phenomena and the present case of unstable systems.

To begin with we introduce a simple model which classically exhibits tachyonlike motion. Consider a real scalar field $\varphi(x,t)$ in one spatial dimension, under the Hamiltonian

$$H = \frac{1}{2}\int dx[\pi^2 + (\partial_x\varphi)^2 + \frac{2m^2}{\lambda}\cos\sqrt{\lambda}\varphi]. \quad (1)$$

Here, $\pi(x,t) = \dot{\varphi}(x,t)$ is the field conjugate to $\varphi(x,t), \lambda > 0$, and the speed of light is put equal to one. This Hamiltonian describes the continuum limit of coupled pendula. We will be interested in the dynamics of φ near the metastable state $\varphi(x) = 0$. For any finite time interval of interest, we make take λ sufficiently small such that the potential term may be expanded, $\cos\sqrt{\lambda}\,\varphi \approx 1 - \frac{\lambda}{2}\varphi^2$. From now on we restrict ourselves to this expansion. Then, the equation of motion for φ becomes

$$\Box\varphi - m^2\varphi = 0. \quad (2)$$

In term of the eigenmodes the solution is

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}}\int dk\, e^{ikx}\varphi_k(t), \quad (3)$$

$$\pi(x,t) = \frac{1}{\sqrt{2\pi}}\int dk\, e^{ikx}\pi_k(t), \quad (4)$$

where reality conditions imply $\varphi_{-k} = \varphi_k^\dagger, \pi_{-k} = \pi_k^\dagger$. The time evolution of the modes is given by

$$\begin{aligned}\varphi_k(t) &= \varphi_{k0}\cos\omega_k t + \frac{\pi_{0k}}{\omega_k}\sin\omega_k t,\\ \pi_k(t) &= \pi_{0k}\cos\omega_k t - \omega_k\varphi_{0k}\sin\omega_k t,\end{aligned} \quad (5)$$

where $\omega_k^2 = k^2 - m^2$. Notice that for $|k| > m, \omega_k$ is real, and φ_k, π_k oscillate in time. In the range $|k| < m$, ω_k is imaginary, and $\varphi_k(t)$ and $\pi_k(t)$ are exponentially diverging. The latter modes are analogous to spontaneous emission in the optical model of an inverted medium of

 0031-9007/98/81(11)/2190(4)$15.00 © 1998 The American Physical Society

two-level systems. We will henceforth refer to the free oscillatory modes and diverging modes as the "normal" and "unstable" modes, respectively. It is only near the point $\varphi \sim \pi \sim 0$ that these modes exist as linearly independent solutions. As φ grows sufficiently, the instability is damped by the nonlinear λ term in (1), but for the time of interest to us, the latter can be neglected.

We now turn to examine several features of the classical equation of motion (2). First, we examine the propagation of a wave packet given by

$$\varphi(x,t) = \int_{|k|>m} dk\,[g_0(k)e^{ikx-i\omega_k t} + \text{H.c.}], \quad (6)$$

and $\pi(x,t) = \partial_t \varphi(x,t)$. We take $g_0(k)$ to be nonzero only in the range of *normal* modes $m < |k| < k_{\max}$, and centered around $k = k_0$, with a width $\Delta k \ll k_0 - m$. The spatial width of this wave packet is $1/\Delta k$, and we assume that at $t = 0$ it is spatially centered far to the left of the origin, at $X_0 \ll 0$, such that only a small tail extends to the region $x > 0$.

This wave packet propagates with a group velocity, v_g, and a phase velocity, v_p, given by

$$v_g = \frac{1}{v_p} = \frac{k_0}{\sqrt{k_0^2 - m^2}}\,. \quad (7)$$

Since $v_g > 1$, the motion of the center of the wave packet (group velocity) is superluminal (tachyonlike), while the phase velocity is always subluminal. As long as the dispersion is negligible, $\varphi(x,t)$ is (up to a time dependent phase factor) just $\varphi(x - v_g t, 0)$, i.e., the initial wave packet moving at velocity v_g.

Second, we note that despite the superluminal group velocity, causality is maintained [2,5]. This is best seen through the Green functions of Eq. (2). In terms of the homogeneous Green functions: $\varphi(x,t) = \int dx'[G(x-x',t)\varphi_0(x') + \tilde{G}(x-x',t)\pi_0(x')]$. However, $\tilde{G}$ and G vanish outside the light cone, i.e., for $x - x' > t$. Therefore *the value of φ at point x at time t is fully determined by its value at points x' that are causally connected to x*.

For further insight into this point, it is instructive to examine what happens if the initial wave packet is truncated at $x = 0$, i.e., $\varphi_0(x)$ and $\pi_0(x)$ are replaced by $\varphi_0^T = \tilde{\theta}(x)\varphi_0(x)$ and $\pi_0^T = \tilde{\theta}(x)\pi_0(x)$, respectively. Here $\tilde{\theta}$ is a smoothed step function. The length scale over which it is smoothed is assumed smaller than all other length scales in the problem and is kept finite just to avoid an infinite $\partial_x\varphi$. Note that since $\varphi_0(x)$ is centered at $X_0 \ll 0$, the truncated wave packet $\varphi_0^T(x)$ constitutes only a small tail of the original one, and the energy stored in the truncated wave packet is just a small fraction of that stored in the original one. The momentum representation of the truncated wave packet $\varphi_0^T(x)$ is $g_0^T(k) = \int dk' \frac{1}{k-k'+i\eta}\, g_0(k')$, with η an infinitesimal number. The smoothing of the step function is accounted for by an upper cutoff to the k integral. While the original wave packet was solely composed of normal modes, the truncated one also includes unstable components.

The time evolution of the truncated wave packet can be calculated using the exact expression for the Green function [2,3,5]. There are three regimes: (i) For points $x < -t$, the causality of the Green function dictates that the value of $\varphi_T(x,t)$ is not affected by the existence of the truncated wave packet at $x > 0$. Thus, the field is zero in this range. (ii) For $-t < x < t$, however, the time evolution of the truncated wave packet is very different from that of the original one: it is exponentially growing due to the contributions of unstable components. Since the amplitude of the normal modes oscillates in time while the amplitude of the unstable modes grows exponentially, for times $t \gg 1/m$ the wave packet $\varphi^T(x,t)$ is predominantly composed of unstable modes of very small wave vectors, with an exponentially small contribution of normal modes. (iii) Finally, for points $x > t$, the causality of the Green function dictates that the value of $\varphi_T(x,t)$ does not depend on whether or not it is truncated at the origin at $t = 0$. It is given, up to a time dependent phase factor, by $\varphi(x - v_g t, 0)$. The propagation of the superluminal peak is not affected by the truncation, and, for long enough times, one finds at $X_0 + v_g t$ a wave packet of width $1/\Delta k$, in which the field φ oscillates with a characteristic wave vector of k_0. Thus, at long times the field $\varphi^T(x,t)$ shows an interesting behavior: it is predominantly composed of very small wave vectors $|k| < m$ with just an exponentially small contribution of wave vectors $|k| > m$. *However, in a region of spatial width $1/\Delta k$ around $X_0 + v_g t$ it oscillates at the large wave vector $k_0 > m$. The region $1/\Delta k$ can be made arbitrarily large, if t is taken long enough.* These short-wavelength oscillations of a long-wavelength wave packet are very similar to the superfast Fourier oscillations discussed in [6–8].

The above discussion demonstrates that in order to maintain consistency with causality, the mechanism which gives rise to superluminal group velocities has to rely on the local information stored in the tail. As long as the true information that the wave has been truncated has not arrived, the local amplification extrapolates the peak as if this truncation does not exist. Classically, extrapolating the full wave from an infinitesimally small tail is, although surprising, possible. As discussed in [5], for an analytic function, the wave can be reconstructed by means of local Taylor expansion, and the tachyonlike propagation can be viewed as an analytic continuation of the tail.

As we now show, the extrapolation of the full propagated wave packet from the truncated tail is made possible by the unstable modes, and this is true independent of the details of the model. Since $g_0^T(k)$ is the Fourier transform of a small tail, it is very small. The propagated wave packet, after truncation, is

$$\varphi^T(x,t) = \int dk[\,e^{ikx-i\omega_k t} g_0^T(k) + \text{H.c.}]\,. \quad (8)$$

However, while on the right-hand side, the amplitudes $g_0^T(k)$, originating from the truncated tail, are very small; on the left-hand side the magnitude of the tachyonic peak is not small, since the peak is fully reconstructed from the tail. Since the amplitude of the stable modes, $\varphi_T(k)$, for $|k| > m$, is constant in time, this amplification can arise only from the contribution of unstable modes growing exponentially with time. It is then essential to have unstable modes for a full reconstruction of a superluminal peak from a truncated tail.

We now proceed to the quantum mechanical analog of this classical system. Can a similar mechanism of local amplification give rise to superluminal group velocities? If so, a process analogous to the classical analytical continuation may use the local information to reconstruct the wave packet's peak.

The causality of the Hamiltonian (1) is manifested in the quantum case by the statement that causally disconnected local observables commute. This causality motivates us to study the quantum analog of the classical truncated wave packet. We first define an analog of a false vacuum state for this model, on top of which we define a quantum state describing a truncated wave packet. Then, we show that if the truncated wave packet is too small (in a sense explained below), the initial quantum state describing it is not orthogonal to the false vacuum state. In that case, the vacuum quantum fluctuations in the fields φ, π are, at $t = 0$, larger than the amplitude of the truncated wave packet, and one cannot distinguish between the false vacuum state and the wave packet state. As we saw above, classically the small truncated wave packet becomes exponentially large in time, and one may naively hope that quantum mechanically it eventually becomes larger than the vacuum fluctuations. However, this cannot be the case, due to the unitary quantum mechanical time evolution: the scalar product of two states, here the false vacuum and truncated wave packet states, is constant in time. Thus, if at $t = 0$ the truncated wave packet is not orthogonal to the false vacuum, it cannot be distinguished from it at a later time. The exponential growth of the tail is masked by an exponential growth in the quantum fluctuations.

Turning to a more concrete discussion, we promote the functions $\varphi(x,t), \pi(x,t)$ to quantum operators. From the standard canonical commutation relations for the fields, $[\varphi(x,t), \pi(x',t)] = i\hbar\delta(x - x')$, it follows as usual that for the normal modes $[\varphi_k, \pi_{k'}^\dagger] = [\varphi_k^\dagger, \pi_{k'}] = i\hbar\delta(k - k')$. However, for the unstable modes the system does not admit the usual Fock space structure with an ordinary particle interpretation. For these modes ($|k| < m$), the Hamiltonian, $\int dk\,[\pi_k^\dagger \pi_k - (m^2 - k^2)\varphi_k^\dagger \varphi_k]$, is unbounded from below and has a continuous and unbounded spectrum of energies for each wave number k. Consequently, these modes cannot be put in a ground state, and the fluctuations in the fields φ_k, π_k inevitably grow exponentially with time.

For the normal modes $|k| > m$ the vacuum state is obviously the ground state of an harmonic oscillator of frequency $\sqrt{k^2 - m^2}$. The situation is less obvious for the unstable modes. On one hand, we would like the quantum fluctuations in the fields φ_k, π_k to be as small as possible, but, on the other hand, stationary states of the unstable modes necessarily have infinite fluctuations. Any state of finite fluctuations in nonstationary, with the fluctuations growing exponentially in time. Thus, we choose the initial state of the unstable modes to be a nonstationary one, in which the exponential growth of the field fluctuations is slowest. This state is the ground state of an harmonic oscillator of frequency $\sqrt{m^2 - k^2}$. The vacuum state of the entire system is then a direct product of the vacuum state for each mode k.

Next, we should find the quantum analog of the wave packet we analyzed in the classical case. Again, causality dictates that a measurement of the field φ at a given point cannot be affected by its values in causally disconnected points. Thus, to analyze the quantum analog of the classical superluminal propagation, we need to construct the quantum analog of the truncated wave packet $\varphi_0^T(x)$. We do so by shifting each mode φ_k from its vacuum state, in which $\langle\varphi_k\rangle = 0$, to a coherent state, in which $\langle\varphi_k\rangle = g_0^T(k)$:

$$|\Psi_0\rangle = \exp\left[\frac{i}{h}\int dk\, g_0^T(k)\hat{\pi}_k^\dagger\right]|\mathrm{vac}\rangle\,, \qquad (9)$$

where $|\mathrm{vac}\rangle$ is the vacuum state. Classically, as we saw above, the truncated wave packet included all the information needed to reconstruct the superluminal propagation of $\varphi_0(x)$. *Quantum mechanically, however, this is true only if the state* (9) *is orthogonal to the vacuum state.* Otherwise, one cannot distinguish between the time evolution of the vacuum state and that of (9).

The scalar product $\langle\mathrm{vac}\,|\Psi_0\rangle$ can be easily calculated,

$$\langle\mathrm{vac}\,|\,\Psi_0\rangle = \exp -\frac{1}{2\hbar}\int dk\,|\omega_k|\,[g_0^T(k)]^2. \qquad (10)$$

There are normal ($|k| > m$) and unstable ($|k| < m$) contributions to the k integral in (10). The physical interpretation of the normal contributions is rather clear: $g_0^T(k)$ is the amplitude of the oscillations of the mode k, and $\frac{\omega(k)}{2\hbar}[g_0^T(k)]^2 dk$ is the average number of photons in that wave vector range. The contribution of the unstable modes cannot be discussed in terms of the photons. For these modes, the integrand $\frac{1}{\hbar}\,|$ is just the ratio of the energy stored in the k mode to $\hbar\omega(k)$. *The quantum state describing the wave packet, $|\Psi_0\rangle$, is then orthogonal to the vacuum state only if the truncated wave packet contains at least several photons of the normal modes or large enough energy in the unstable modes.* This poses a quantum mechanical condition for the reconstruction of a tachyonic peak from a truncated tail: the latter cannot be too small. If it is too small, the initial quantum fluctuations in the field φ, which grow exponentially with time, overcome the reconstructed peak. This observation limits the superluminal propagation of the wave packet's peak: for time t the wave packet can be reconstructed at a point X_0

only if at time $t = 0$ the tail in the region $x > X_0 - t$ is large enough to be orthogonal to the vacuum.

To exemplify the last point, let us evaluate the fluctuations of a local observable. The field at a point has singular fluctuations, and one needs to consider smeared operators like $\overline{\varphi}_{(L,t)} \equiv \int f(x')\varphi(L - x', t)\, dx'$, where $f(x')$ is nonvanishing only within a distance ΔL around the point L, and satisfies $\int f dx = 1$. Since $\langle \Psi_0 | \overline{\varphi} | \Psi_0 \rangle$ is identical to the classically smeared field, the fluctuations are dominated by $\langle \Psi_0 | \overline{\varphi}^2 | \Psi_0 \rangle = \langle \overline{\varphi}^2 \rangle_N + \langle \overline{\varphi}^2 \rangle_U$, where the subscripts N and U stand for the contributions of the normal and unstable modes. The contribution of the normal modes is always finite. The unstable modes, however, yield an exponentially growing fluctuation

$$\langle \overline{\varphi}^2 \rangle_U \approx \hbar I_0[2mT] \xrightarrow{T \gg 1/m} \hbar \frac{e^{2mT}}{\sqrt{4\pi m T}}, \qquad (11)$$

where I_0 is the zeroth order modified Bessel function. For the observation of superluminal propagation two conditions have to satisfied: First $v_g T_{\rm obs} \gg \frac{1}{\Delta k}$ (where $T_{\rm obs}$ is the time at which the wave packet is observed); i.e., the point of observation should be far outside the initial spread of the wave packet. Second, we must distinguished between superluminal propagation and propagation at the speed of light. This leads to $(v_g - 1)T_{\rm obs} \gg \frac{1}{\Delta k}$. These conditions require that $T_{\rm min} m \gg 1$. Equation (11) thus implies that for the signal amplitude to be larger than the amplitude of the fluctuations at the observation time, the signal amplitude should be exponentially large.

One may question whether our assumption that the unstable modes are in a direct product state limits the generality of our result. We now argue that the obstacles pointed out above in the way of realizing quantum mechanical superluminal group velocities cannot be circumvented by changing that assumption. Consider three classical tachyonlike wave packets, denoted by R, L, and D, with initial ($t = 0$) analytic field and momenta configurations (φ_R, π_R), $(\varphi_L = \varphi_R, \pi_L = -\pi_R)$, and $(\varphi_D = -\varphi_R, \pi_D = \pi_R)$. Here R and L are two wave packets for which φ is identical at $t = 0$ but whose direction of motion is opposite, being right for R and left for L. The third wave packet, D, moves to the right but has an initial φ opposite to that of R. Suppose that initially the wave packets are localized around $X_0 \ll 0$. Then after a long enough time the right moving wave packets R and D form peaks at $X \gg 0$. The left moving one, L, forms a peak at $X \ll 0$ and leaves at $X \gg 0$ only a small tail. Consider again the truncated tails of the three wave packets, denoted by R^T, L^T, and D^T, where the truncation is done at $X = t = 0$. Classically, the time evolution of all three wave packets in the region $X > t$ is not affected by the truncation. *Quantum mechanically, however, it might not be possible for all three quantum states to be mutually orthogonal:* R^T and L^T are orthogonal if their momentum uncertainty is smaller than the difference between their momentum expectation values: $\Delta \pi^T_{(L,R)} \ll |\langle \pi^T_R \rangle - \langle \pi^T_L \rangle|$. Similarly, R^T and D^T are orthogonal only if $\Delta \varphi^T_{(R,D)} \ll |\langle \varphi^T_L \rangle - \langle \pi^T_R \rangle|$. However, if the tail is made arbitrarily small, at least one of the two inequalities must be violated. This is so since the right-hand side of the two inequalities must approach zero as the tail is made smaller and smaller, while the left-hand side cannot be made arbitrarily small due to the uncertainty principle. Thus, if the tails are too small, the wave packets R^T, L^T, and D^T, which are classically distinguishable, cannot be mapped into three mutually orthogonal quantum states. Quantum fluctuations then make it impossible to reconstruct the full wave packet from its tail. Because of the unitary evolution of quantum states, the scalar product of two quantum states is time independent. Thus, the limitations found above to the reconstruction of quantum wave packets from their causal tails cannot be overcome by an amplification of the tails.

In conclusion, we have shown that classical superluminal-like effects become incompatible with unitarity in the quantum mechanical limit and are strongly suppressed. Their suppression results from the smallness of the wave packet's tail, in a similar way to the suppression of superluminal effects in tunneling. Our argument strongly questions the possibility that these systems may have tachyonlike quasiparticle excitations made of a small number of photons [4].

Y. A. acknowledges the support of the Basic Research Foundation, Grant No. 614/95, administered by the Israel Academy of Sciences and Humanities. A. S. acknowledges support of the U.S.–Israel Bi-national Science Foundation (No. 95/250-1), the Minerva foundation, and the V. Ehrlich career development chair.

[1] For a recent review see R. Y. Chiao and A. M. Steinberg, *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1998), Vol. 37.
[2] Y. Aharonov, A. Komar, and L. Susskind, Phys. Rev. **182**, 1400 (1969).
[3] R. Y. Chiao, Phys. Rev. A **48**, R34 (1993); R. Y. Chiao and J. Boyce, Phys. Rev. Lett. **73**, 3383 (1994).
[4] R. Y. Chiao, A. E. Kozhekin, and G. Kurizki, Phys. Rev. Lett. **77**, 1254 (1996); G. Kurizki, A. Kozhekin, and A. G. Kofman (to be published).
[5] G. Diener, Phys. Lett. A **223**, 327 (1996).
[6] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[7] M. V. Berry, J. Phys. A **27**, L391 (1994).
[8] B. Reznik, Phys. Rev. D **55**, 2152–2158 (1997).