On the Bohr-Rosenfeld measurability problem for non-linear systems

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Abstract. The singular nature of fields at a space-time point, forces us to consider space-time weighted (smeared) field distributions as elementary observables of field theory. As a consequence, measurement theory for fields, must handle the non-trivial task of eliminating the back-reaction disturbance caused by the measurement process itself. For the case of a free field, Bohr and Rosenfeld showed that an additional spring-like mechanism can compensates this disturbance. In this article we attempt to extend their results to non-linear systems. We construct a perturbative extension of the Bohr-Rosenfeld compensation procedure. We show however that this method breaks down for sufficiently accurate measurements and leads to a lower bound on the precision of the measurement. The failure of the perturbative approach raises again the question of measurability for non-linear systems.

Keywords: quantum measurements, foundations of quantum theory

Mathematics Subject Classifications (2000): 81P15, 81B10

It is indeed a very sad occasion when we realize that Moshé is not with us anymore. I first heard about Moshé many years ago from my neighbor when I was in high school. It turned out that this neighbor was adopted by Moshé's parents when she came as an orphan to Israel. She was full of praise for their generosity and she told me that they had a son Moshé who was a mathematical genius, and that I should try to meet with him. That meeting took place years later in New York. I was immediately immensely impressed by his enthusiasm and his brilliant ideas. Since then we met many times and became close friends. It is still difficult for me to believe that we will never meet again. His untimely death is a great loss to all of us.

Y. Aharonov

1. Introduction

The measurement of an observable **A** is ideally described by an impulsive coupling between the system and the measuring device at a certain instant of time [1]. However in realistic measurements the duration of the coupling between the systems and the measuring device is finite. In this case the interaction with the measuring device modifies the evolution of the system and consequently instead of observing

G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 1, 99 – 108.

the undisturbed time average

$$\bar{\mathbf{A}} = T^{-1}\int_0^T \mathbf{A}(t')dt' \tag{1}$$

we observe a disturbed time average which also depends on the initial state of the measuring device.

This problem becomes particularly acute in the context of quantum field theory. The singular nature of fields at a point forces us to consider instead space-time weighted (smeared) fields as elementary observables of quantum field theory [2]. Therefore, in this case we have to deal on a fundamental level with the problem of measuring space-time averaged observables. Measurement theory for fields, must hence handle the non-trivial task of eliminating the back-reaction disturbance caused by the measurement process itself. Indeed Bohr and Rosenfeld showed that in addition to the usual measurement interaction, one can introduce a spring-like mechanism which compensates this disturbance and allows a precise observation of the undisturbed space-time average [3] . However their work was restricted to the special case of a non-interacting linear system.

In this article we will attempt to extend their results to non-linear systems. As a toy model we shall consider the simpler case of non-relativistic non-linear harmonic oscillator. This model simplifies the problem because we do not have to deal with spatial averages, but nevertheless it still contains the essential features of the full problem. We then consider problem of measuring a time averaged observable by means of a perturbative approach in the non-linear coupling constant. As we will show the validity of a perturbative method breaks down for sufficient accurate measurement and gives rise to a lower bound on the precision of the measurement.

2. Non-linear harmonic oscillator as a toy problem

We shall consider as a toy model for a non-linear system a harmonic oscillator with a potential given by

$$H = \frac{1}{2}(\mathbf{p}^2 + \Omega^2\mathbf{x}^2) - \frac{\lambda}{n}\mathbf{x}^n, \tag{2}$$

where $n > 2$.

Our first aim will be to describe a measurement of an averaged observable such as

$$\bar{\mathbf{x}} = \frac{1}{T}\int_0^T \mathbf{x}(t')dt'. \tag{3}$$

In the limit of $T \to 0$ the prescription is know: The appropriate interaction term is in this case

$$H_I = -g(t)\mathbf{Q}\mathbf{x}, \tag{4}$$

where $\mathbf{Q}$ is canonically conjugate to the "output variable" $\mathbf{P}$, and $g(t) = g_0\delta(t)$. It is assumed that the effective mass of the device is infinitely large, thus the kinetic part of the measuring device Hamiltonian vanishes and $\mathbf{Q}$ is a constant of motion. The interaction Hamiltonian (4) yields:

$$\delta\mathbf{P} \equiv \mathbf{P}(+\varepsilon) - \mathbf{P}(-\varepsilon) = g_0\mathbf{x}(0). \tag{5}$$

As a starting point let us modify this interaction by replacing the Dirac delta function by a smooth function $g(t)$ with a compact support in the time interval $0 < t < T$.

The solution to the equations of motion

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{p} \\ \dot{\mathbf{p}} &= -\Omega^2\mathbf{x} + \lambda\mathbf{x}^{n-1} + g(t)\mathbf{Q} \\ \dot{\mathbf{P}} &= g(t)\mathbf{x} \end{aligned} \tag{6}$$

can be written in the integral form:

$$\mathbf{x}(t,g) = \mathbf{x}_0(t) + \int_0^t \frac{F(t')}{\Omega}\sin[\Omega(t-t')]dt' \tag{7}$$

$$\mathbf{p}(t,g) = \mathbf{p}_0(t) + \int_0^t F(t')\cos[\Omega(t-t')]dt', \tag{8}$$

where

$$F(t) = \lambda\mathbf{x}^{n-1}(t,g) + g(t)\mathbf{Q}, \tag{9}$$

and $\mathbf{x}_0(t)$, $\mathbf{p}_0(t)$ are free solutions ($\lambda = g = 0$), which coincide with the non-linear solution $\mathbf{x}(t)$ at $t = 0$. The presence of the coupling parameter g in $\mathbf{x}(t,g)$ will be used in the following to denote an explicit dependence of the solution on variables of the measuring device.

The solutions $\mathbf{x}(t,g)$ and $\mathbf{p}(t,g)$ depend on the back-reaction via the term $g(t)\mathbf{Q}$. Contrary to classical mechanics, due to the uncertainty principle, we cannot make $\mathbf{Q}$ as small as we wish and still obtain an accurate measurement, i.e., $\Delta\mathbf{P}(0) \to 0$. Consequently, we finally observe

$$\mathbf{P}(T) - \mathbf{P}(0) = \int_0^T g(t)\mathbf{x}(t,g)dt, \tag{10}$$

which is not the undisturbed (g=0) value of $\bar{\mathbf{x}}$. Only in the limiting case $T = 0$ does the error vanish.

In the next section we show how the undisturbed $\bar{\mathbf{x}}$ can be observed in the linear case: $\lambda = 0$.

3. The linear case

We shall now proceed to examine the case of a measurement of an averaged observable and show how to eliminate the back reaction in this special case.

For simplicity we shall choose the weight function as $g(t) = g_0$ for $t \in (0,T)$ and zero otherwise. In this case, the solution of the equations of motion when $\lambda = 0$ is

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0(t) + \frac{g_0(1-\cos\Omega t)}{\Omega^2}\mathbf{Q} \\ &\equiv \mathbf{x}_0(t) + \alpha(t)\mathbf{Q}.\end{aligned} \tag{11}$$

Thus, the $\mathbf{P}$ coordinate of the measuring device will be shifted by

$$\begin{aligned}\delta\mathbf{P} &= \mathbf{P}(T) - \mathbf{P}(0) \\ &= g_0\int_0^T \mathbf{x}(t')dt' = g_0\int_0^T \mathbf{x}_0(t')dt' + \mathbf{Q}\frac{g_0^2 T}{\Omega^2}\left(1 - \frac{\sin\Omega T}{\Omega T}\right).\end{aligned} \tag{12}$$

The undisturbed average $\bar{\mathbf{x}} = \bar{\mathbf{x}}_0 = T^{-1}\int_0^T \mathbf{x}_0(t')dt'$ is given by

$$\bar{\mathbf{x}}_0 = \frac{\delta\mathbf{P}}{g_0 T} - \frac{g_0}{\Omega^2}\left(1 - \frac{\sin\Omega T}{\Omega T}\right)\mathbf{Q}. \tag{13}$$

Since $\bar{\mathbf{x}}_0$ depends linearly on both $\mathbf{P}$ and $\mathbf{Q}$, a precise measurement of $\mathbf{P}$ causes a larger uncertainty in the second term above, i.e., it increases the back reaction of the measuring device on the oscillator. Since the uncertainty in $\bar{\mathbf{x}}$ is

$$\Delta\bar{\mathbf{x}}_0 \approx \frac{\Delta\mathbf{P}}{g_0 T} + \frac{g_0}{\omega^2\Delta\mathbf{P}}\left(1 - \frac{\sin\Omega T}{\Omega T}\right), \tag{14}$$

the minimal uncertainty is

$$\Delta\bar{x}_{min} = \frac{2}{\Omega}\sqrt{1 - \frac{\sin\Omega T}{\Omega T}} \overset{T\to 0}{\to} \sqrt{\frac{2}{3}}T, \tag{15}$$

which vanishes only in the impulsive limit. We also note that in the limit of $T \to \infty$ the disturbance does not average out but rather approaches the constant $\frac{2}{\Omega}$.

The direct approach therefore fails to measure $\bar{\mathbf{x}}_0$ precisely. There are however ways to correct or eliminate the error above, and in the following we present three different ways to achieve this goal.

The idea of Bohr and Rosenfeld was to correct the error by adding to the Hamiltonian (4) a new "compensating" term. The error in the shift of $\mathbf{P}$ in (12) appears as linear in $\mathbf{Q}$, very much like the effect of a spring in the $\mathbf{Q}$-coordinates. Since the coefficient which multiplies $\mathbf{Q}$ in (12) is known, it is straightforward to compensate for this error by adding to the interaction Hamiltonian (4) a spring term:

$$H_I = -g(t)\mathbf{Q}\mathbf{x} + \frac{1}{2}k\mathbf{Q}^2, \tag{16}$$

where k can be chosen for example as

$$k(t) = \frac{g^2(t)T}{\Omega^2}\left(1 - \frac{\sin\Omega T}{\Omega T}\right). \tag{17}$$

It is straightforward to see that the new spring-term in the equations of motion precisely eliminates the back-reaction of the device on the system. In the limit $T \to 0$, the compensation is of course not needed since the spring constant vanishes, and we obtain back the ordinary impulsive measurement.

Another approach due to Unruh [4], does not require modification of the interaction (4) but requires instead an additional measuring device. Inspecting Equation (13), we notice that $\bar{\mathbf{x}}_0$ is given on the right-hand side in terms of a linear combination of $\mathbf{P}$ and $\mathbf{Q}$. Therefore, after preparing the measuring device in an initial state with a well-defined $\mathbf{P}$ one can measure by means of *another* measuring device the linear combination

$$\frac{1}{g_0 T}\mathbf{P} - \frac{k(T)}{g_0}\mathbf{Q}(T), \tag{18}$$

where k is given in (17). As before this method requires that the back-reaction effect on the system depends only on variables of the measuring device. As we shall see in Section 4, both methods apparently fail when non-linearities introduce back-reaction terms which depend also on the system itself.

A third different method to measure $\bar{\mathbf{x}}$ could be the following. By integrating $\mathbf{x}(t)$ we can express $\bar{\mathbf{x}}$ as

$$\frac{1}{T}\int_0^T (\mathbf{x}_0 \cos\Omega t' + \frac{1}{\Omega}\mathbf{p}_0 \sin\Omega t')dt' = \frac{\mathbf{x}_0}{\Omega T}\sin\Omega T + \frac{\mathbf{p}_0}{\Omega^2 T}(1 - \cos\Omega T). \tag{19}$$

Note that $\mathbf{x}_0$ and $\mathbf{p}_0$ are constants of motion. By inverting the solutions for the equations of motion for $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{p}_0)$ and $\mathbf{p} = \mathbf{p}(\mathbf{x}_0, \mathbf{p}_0)$ we obtain

$$T\bar{\mathbf{x}}(t) = \frac{2}{\Omega}\sin\Omega T/2\left[\mathbf{x}(t)\cos\Omega(T/2 - t) + \frac{1}{\Omega}\mathbf{p}(t)\sin\Omega(T/2 - t)\right]. \tag{20}$$

Thus, we have expressed $\bar{\mathbf{x}}$ in terms of a (time dependent) constant of motion. We can now set up a continuous measurement

$$H_I = -g(t)\mathbf{Q}\bar{\mathbf{x}}(t). \tag{21}$$

Clearly the observable $\bar{\mathbf{x}}(t)$ remains a constant of motion and is not disturbed by the interaction. We have thus managed to measure $\bar{\mathbf{x}}$ by a continuous measurement of a constant of motion whose value is identical to $\bar{\mathbf{x}}$ for any t.

4. Non-linear case

The presence of non-linear terms makes the back-reaction dependent on both system and the measuring device. Let us now show that the methods of Section 3, which are directly based the idea of constructing a compensation are likely to fail. Let us assume again the exact solution is given, and let us express it as

$$\mathbf{x}(t,g) = \xi(t,\lambda,g=0) + \Delta\xi(t,\lambda,g). \tag{22}$$

Here, $\Delta\xi(t,\lambda,g) = \xi(t,\lambda,g) - \xi(t,\lambda,g=0) = x(t,g) - x(t,g=0)$ is the non-linear error due to the measurement. Thus, using the naive coupling (4) we obtain:

$$\frac{\delta \mathbf{P}}{g_0 T} = \bar{\mathbf{x}} + \frac{1}{g_0 T}\int_0^T \Delta\xi(t,\lambda,g)dt'. \tag{23}$$

To proceed, we would like to construct a compensation to the last term. Thus, $\Delta\xi$ must be expressed in terms of the dynamical variables $\mathbf{x}(t,g)$, $\mathbf{p}(t,g)$ and $\mathbf{Q}$. However, $\Delta\xi$ depends on $\mathbf{x}(t,g=0)$ and we first need to find the non-linear relation between the disturbed non-linear $\mathbf{x}(t,g)$ and the undisturbed $\mathbf{x}(t,g=0)$ solutions:

$$\mathbf{x}(t,g) = f[\mathbf{x}(t,0),\mathbf{p}(t,0),\mathbf{Q}], \tag{24}$$

etc. The latter equation corresponds to a "canonical" transformation between two non-linear solutions of two different systems. It is not clear if such a well-defined relation indeed exist. Assuming however that the transformation above does exist, we may now attempt, as before, to add to the Hamiltonian the compensation term $\int[k\mathbf{Q}+\Delta\xi(\lambda,g)]dq$. However, the new equations of motion will give rise to another non-linear error $\Delta\xi' \neq \Delta\xi$. Therefore, a self-consistent scheme for constructing a compensation must be found, which deals with the non-perturbative effects of the compensation on the system.

The difficulties in finding an exact non-perturbative compensation scheme, leads us to examine a more modest approach. It can be hoped that at least for small averaging time T, the effect of the non-linearities is small. Therefore, in the next section, we shall examine a perturbative approach.

5. A perturbative approach

For a small non-linear constant, λ, we shall attempt a perturbative approach. As a starting point we shall assume that the measurement can be expressed by:

$$H_I = -g(t)\mathbf{Q}\mathbf{x} + H_C, \tag{25}$$

i.e., as a sum of the naive measurement and a compensation term. Clearly in the limit of $T \to 0$, H_I should reduce to an ordinary impulsive measurement, i.e., $H_C(T=0)=0$. Our aim will be to construct a self-consistent procedure to compute the compensation term H_C to any order in λ:

$$H_C = \Delta H^{(0)} + \sum_k \lambda^k \Delta H^{(k)}. \tag{26}$$

The terms $\Delta H^{(k)}$ are the analogous compensations to the back-reaction up to the k'th order in λ.

To this end, let us expand the solution for $\mathbf{x}(t)$ as

$$\mathbf{x}(t,g) = \mathbf{x}^{(0)} + \lambda \mathbf{x}^{(1)}(t) + \lambda^2 \mathbf{x}^{(2)}(t) + \cdots \tag{27}$$

and at each order separate the undisturbed solution, denoted as $\mathbf{x}_0^{(k)}$, from the non-linear error, denoted as $\Delta x^{(k)}$:

$$\mathbf{x}^{(k)}(g) = \mathbf{x}_0^{(k)}(t) + \Delta \mathbf{x}^{(k)}(t,g), \tag{28}$$

i.e., for $g = 0$ we have $\Delta \mathbf{x}^{(k)}(t, g = 0) = 0$ and $\mathbf{x}^{(k)}(g = 0) = \mathbf{x}_0^{(k)}(t)$ is the solution without the coupling to a measuring device.

Likewise, the shift of the output register, $\delta \mathbf{P} = \mathbf{P}(T) - \mathbf{P}(0)$, will be expanded in powers of λ, and at each order we shall evaluate the terms of the decomposition:

$$\delta \mathbf{P}^{(k)} = \delta \mathbf{P}_0^{(k)} + \Delta \mathbf{P}^{(k)}. \tag{29}$$

Here, $\Delta \mathbf{P}^{(k)}$ is the error due to the back reaction to the k'th order. The knowledge of $\Delta \mathbf{P}^{(k)}$ will allow us to construct the appropriate compensation $\Delta H^{(k)}$, up to the same order k.

To proceed we shall now simplify the problem, and let $n = 3$ in (2). It can be shown however that the following will be valid for any general non-linear potential.

To evaluate $\Delta \mathbf{P}^{(k)}$ we first use (7) to obtain an integral solution to any order:

$$\mathbf{x}^{(0)}(t,g) = \mathbf{x}_0(t) + \mathbf{Q} \int_0^t g(t')D(t-t')dt', \tag{30}$$

$$\mathbf{x}^{(1)}(t,g) = \int_0^t \mathbf{x}^{(0)}(t')\mathbf{x}^{(0)}(t')D(t-t')dt', \tag{31}$$

$$\mathbf{x}^{(2)}(t,g) = 2\int_0^t \mathbf{x}^{(0)}(t')\mathbf{x}^{(1)}(t')D(t-t')dt', \tag{32}$$

etc., where $\mathbf{x}_0$ is the free solution ($g = \lambda = 0$), and

$$D(t) = \frac{\sin \Omega t}{\Omega}. \tag{33}$$

The result for the zero'th order was already obtained above, where we found

$$\Delta H_0 = \frac{1}{2} k \mathbf{Q}^2, \tag{34}$$

where k is the c-number given by (17). Here, and in the following we shall ignore any potential problems with ordering of non-commuting operators.

To the first order in λ we obtain

$$\delta \mathbf{P}^{(1)} = g_0 \int_0^T dt \int_0^t [\mathbf{x}^{(0)}]^2 dt'. \tag{35}$$

Thus,

$$\begin{aligned}
\Delta \mathbf{P}^{(1)} &= g_0 \int_0^T dt \int_0^t \left[2\alpha(t')\mathbf{Q}\mathbf{x}_0(t') + \alpha^2(t')\mathbf{Q}^2\right] dt' \\
&= g_0 \int_0^\infty \left[2\alpha(t')\mathbf{Q}\mathbf{x}_0(t') + \alpha^2(t')\mathbf{Q}^2\right] dt' \int_0^T \theta(t-t')dt \\
&= g_0 \int_0^\infty \Big((T-t')\theta(T-t') - (-t')\theta(-t')\Big) \left[2\alpha(t')\mathbf{Q}\mathbf{x}_0(t') + \alpha^2(t')\mathbf{Q}^2\right] dt' \\
&= g_0 \int_0^T (T-t') \left[2\alpha(t')\mathbf{Q}\Big(\mathbf{x}^{(0)}(t') - \mathbf{Q}\alpha(t')\Big) + \alpha^2(t')\mathbf{Q}^2\right] dt' \\
&= g_0 \int_0^T (T-t') \left[2\alpha(t')\mathbf{Q}\mathbf{x}^{(0)}(t')dt' - \alpha^2(t)(T)\mathbf{Q}^2\right] dt'. \qquad (36)
\end{aligned}$$

where $\alpha(t)$ is defined in (11). As could have been anticipated, the new feature of the error terms up to the first order in λ, is their depends on the variable $\mathbf{x}$ of the system. The last term does not depend on system variables and therefore may be trivially compensated as a spring-like compensation. Alternatively, as discussed in Section 3, we may use a second measuring device to measure the combination $\frac{1}{g_0 T}\mathbf{P} - k\mathbf{Q} - k'(T)\mathbf{Q}^2$, where k' can be found by integrating the last term in (36).

Equation (36) suggests adding as a compensation the term

$$\Delta H^{(1)} = g^2(t)(T-t)\left[\alpha(t)\mathbf{Q}^2\mathbf{x}(t) - \frac{1}{3}\alpha^2(t)\mathbf{Q}^3\right]. \qquad (37)$$

With the compensation $\Delta H^{(0)} + \lambda \Delta H^{(1)}$ added to the Hamiltonian, the equations of motion become

$$\dot{\mathbf{P}} = g(t)\mathbf{x} - k(t)\mathbf{Q} + \lambda g^2(t)(T-t)\left[2\alpha(t)\mathbf{Q}\mathbf{x}(t) - \alpha^2(t)\mathbf{Q}^2\right], \qquad (38)$$

$$\frac{d^2\mathbf{x}}{dt^2} + \Omega^2\mathbf{x} = \lambda\mathbf{x}^2 + g(t)\mathbf{Q} + \lambda g^2(t)(T-t)\alpha(t)\mathbf{Q}^2. \qquad (39)$$

The new term on the right-hand side of (39) modifies the solution for $x(t)$ to

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \alpha(t)\mathbf{Q} + \lambda\beta(t)\mathbf{Q}^2 + \lambda\int_0^T \mathbf{x}^2(t')dt'. \qquad (40)$$

The first order is modified by the $\beta(t)\mathbf{Q}^2$ which can trivially be compensated by adding also the term $\frac{1}{3}\lambda g(t)\beta(t)\mathbf{Q}^3$. This completes the compensation to the first order. It is important to note that the a first order compensation cannot modify lower order compensations but only higher orders. This allows us to go to higher orders in λ without effecting the corrections found at lower orders.

Let us see how one can proceed to higher orders. To the second order we obtain:

$$\frac{d\Delta\mathbf{P}^{(2)}}{dt} = g(t)\mathbf{Q}\Delta\mathbf{x}^{(2)} + 2g^2(t)(T-t)\alpha(t)\mathbf{Q}\Delta\mathbf{x}^{(1)}(t), \qquad (41)$$

where

$$\Delta \mathbf{x}^{(1)}(t) = \beta(t)\mathbf{Q}^2 + \int_0^t \Delta[\mathbf{x}^{(0)}(t')]^2 dt', \tag{42}$$

and

$$\Delta \mathbf{x}^{(2)}(t) = 2\int_0^t \Delta[\mathbf{x}^{(0)}(t')\mathbf{x}^{(1)}(t')]dt' \tag{43}$$

where the notation $\Delta[\cdots]$, on the right-hand side in the equations above, indicates that only **Q**-dependent terms of $[\cdots]$ are included.

The computation can be carried out as before. The only technical subtlety is that now $\Delta \mathbf{P}^{(2)}$ contain also triple integrals over time like:

$$\int dt \int x_0(t')dt' \int p_f(t'')dt''. \tag{44}$$

To proceed one has to reduce this integral to a single integration over time, or eliminate the integrals completely. Otherwise, the compensations would have to be non-local in time. In our case of a first quantized theory, we can always use the property of linear operators: $\mathbf{p}_0(t)$ (or $\mathbf{x}_0(t)$) can always be expressed as a linear combination of $\mathbf{x}_0(t=0)$ and $\mathbf{p}_0(t=0)$. Therefore, we can integrate over free operators and reduce: $\int_0^{t'} \mathbf{p}_0(t'')dt'' = c_1(t')\mathbf{x}_0(t') + c_2(t')\mathbf{p}_0(t')$. The integral above can be hence reduced to a single integral of the form $\int (c_1'(t)\mathbf{x}_0^2 + c_2'(t)\mathbf{x}_0\mathbf{p}_0)dt$.

Finally we find that the compensation to the second order has the following form:

$$\Delta H^{(2)} = \mathbf{Q}^2(\gamma_1\{\mathbf{x},\mathbf{p}\} + \gamma_2\mathbf{x}^2) + \mathbf{Q}^3(\gamma_3\mathbf{x} + \gamma_4\mathbf{p}) + \gamma_5\mathbf{Q}^4. \tag{45}$$

where $\gamma_i(T)$ are generally time dependent c-numbers. In principle it seems that this procedure can be carried out to any order in λ.

6. Break-down of the perturbative approach

Although by using a perturbative approach we can formally evaluate self-consistently compensation terms to any order in the nonlinear coupling constant, this approach still does not allow precise measurements. Unlike classical measurements, in the limit of a precise measurement, the variable **Q**, which is conjugate to the "output register" **P**, becomes completely uncertain. Since the expansion for the compensation H_C generally depends on **Q**, for a given T, the validity of the perturbative expansion must break-down beyond a certain precision $\Delta \mathbf{x}_{min}$.

To show this consider the first order compensation term. From (37) we have $\Delta H^{(1)} \approx T\alpha(T)\mathbf{Q}^2\mathbf{x}$. A necessary (but not sufficient) condition for a well-behaved expansion is therefore:

$$\lambda T\alpha(T)\mathbf{Q}^2\mathbf{x} \sim \lambda T^3\mathbf{Q}^2\mathbf{x} << 1. \tag{46}$$

Substituting a rough approximation $\mathbf{Q}^2 \sim (\Delta \mathbf{Q})^2$ and $\mathbf{x} \sim \Delta x$ and using $\Delta \mathbf{P} \sim \Delta \mathbf{x}/g_0 T$ we obtain the condition

$$\Delta \mathbf{x} > \Delta \mathbf{x}_{min} \approx \lambda g_0^2 T^5. \tag{47}$$

Only by letting $T \to 0$ or $\lambda \to 0$ we regain $\Delta \mathbf{x}_{min} \to 0$.

7. Discussion

In this article we have examined the measurability of observables by means of a continuous measurement for linear and non-linear theories.

We found that a perturbative approach for constructing compensations breaks down for sufficiently accurate measurements. The reason for this failure is that the disturbance at each order generally depends on increasing powers of $\mathbf{Q}$, (the conjugate to the output variable $\mathbf{P}$). For a sufficiently precise measurements, $\Delta\mathbf{Q} \sim 1/\Delta\mathbf{P}$ becomes arbitrarily large. Hence, for any λ, and for any given interaction time T, there exists a sufficiently large $\Delta\mathbf{Q}$ which invalidates the perturbative expansion. Therefore our method gives rise to a lower bound on the accuracy.

Acknowledgements. We thank W. G. Unruh for valuable discussions and L. Vaidman for helpful remarks. We acknowledge the support from grant 471/98, of the Israel Science Foundation, established by the Israel Academy of Sciences and Humanities. The work of Y. A. was also supported by NSF grant PHY-9971005.

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