Combined electric and magnetic Aharonov–Bohm effects

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(Received 17 October 2006; accepted 28 August 2007)

It is well known that the electric and magnetic Aharonov–Bohm effects may be formally described on an equal footing using the four-vector potential in a relativistic framework. We give an illustrative manifestation of both effects in a single configuration in which the path of the charged particle determines the weight of the acquired electric and magnetic relative phases. The phases can be distinctively obtained in the Coulomb gauge. The examples illustrate that, although each of the relative phases is gauge dependent, their sum is gauge invariant.

[DOI: 10.1119/1.2787012]

The Aharonov–Bohm[1] effect is traditionally separated into two effects: the electric effect and the more familiar magnetic one. These effects may be formally described on an equal footing using the four-vector potential in a relativistic framework, where the acquired relative phase is $\oint A^{\mu}dx_{\mu}$. A physical manifestation of their simultaneous existence can be achieved with a simple particular configuration in space-time. In this paper we give such a manifestation, which makes tangible the relation between the two effects.

Both the electric and the magnetic Aharonov–Bohm effects illustrate the significance of electromagnetic potentials on charged particles that travel through regions of space that are not simply connected. Even if such particles encounter no electromagnetic fields in their paths, they may acquire a relative phase resulting in a shift in the interference pattern. In the electric effect, a charged particle's wave packet passes through a non-simply connected region, where it is split in two, and each of the two wave packets encounters a different scalar potential that may depend only on time. Therefore, no electric forces act on the particle. The relative phase acquired is $\oint \phi dt$ in Gaussian units where we take $\hbar = c = e = 1$.

This electric field effect may be physically realized in the following way: A charged particle's wave packet starts on one side of a capacitor and is split so that one of the wave packets travels through the capacitor while the other goes in the opposite direction [see Fig. 1(a)]. The capacitor is initially uncharged. We assume there is a little hole in the capacitor through which the wave packet travels undisturbed. Then the capacitor is charged and discharged, and only afterward do the wave packets return together and interfere (so that one of the wave packets travels through the hole). We assume that the wave packets are very far from the boundaries of the capacitor (or that the capacitor plates are infinite along the x and z axes) and that the hole is small. Therefore, each of the wave packets travels through a region of zero electric field. Instead of charging and discharging the capacitor, we can just move the plates of a charged capacitor. Initially, two oppositely charged plates are zero distance apart so that the electric field vanishes. The capacitor plates are then moved apart and then together again in the y direction. This setup is illustrated in Fig. 1(b).

In the magnetic Aharonov–Bohm effect, a charged particle's wave packet interferes around a confined magnetic flux, acquiring a relative phase of $\oint \mathbf{A}\cdot d\mathbf{x} = \oint \mathbf{B}\cdot d\mathbf{S}$, where we take the magnetic susceptibility and the electric permittivity to be unity. This effect may be physically realized using a long solenoid (see Fig. 2).

Our example is based on constructing a subspace in space-time in which the four-vector potential is non-trivial. Although the sources that yield the suggested potential depend on time, we arrange the setup to be non-radiating so that both the electric and magnetic fields vanish everywhere outside the source. When a charged particle's wave packet encounters such a potential, it acquires a phase $\theta = \oint \mathbf{A}\cdot d\mathbf{l} + \oint \phi dt$ from both the electric and magnetic effects, which depend on the particle's particular path in space-time. By expressing the potential in the Coulomb gauge we can see which part of the phase is electric and which is magnetic. In this way we see how the electric and magnetic relative phases depend on different possible paths.

The setup that combines both effects is shown in Fig. 3. Assume that the two capacitor plates are infinite in the z direction and zero distance apart in the y direction. The plates are finite in the x direction and the edges are located at x_0 and x_1; we define $L = x_1 - x_0$. In addition, two infinitesimally thin and infinitely long solenoids (fluxons) are located on the edges of the capacitor (along the z axis).

The time dependence of the setup is illustrated in Fig. 4. At $t = t_0$ the plates are instantaneously moved apart and then together again. At $t = t_1$ we again apply the same procedure: the plates are instantaneously moved apart and then together, but in the opposite direction (as shown in the small graph in Fig. 4). To cancel the resulting electromagnetic wave fronts, we simultaneously send opposite and steady currents through the two solenoids during the time interval $[0, T]$, where $T = t_1 - t_0$. This choice of sudden changes in the charge and current densities is not essential and is used to simplify the calculations, as shall be explained later. As we now show, the suggested configuration describes a non-radiating source, which creates a non-simply connected topology in space-time. A charged particle that enters this finite volume in the xt plane acquires a phase from both the electric and magnetic effects.

First consider a particle moving along path 1 of Figs. 3 and 4, which we regard as the "electric path:" at $t < t_0$ its wave packet is located at (x, y, z) where $x_0 < x < x_1$, $y < y_0$, and z is arbitrary (and will be omitted from now on). The wave packet then splits in the y direction, so that at $t = t_0$ the two wave packets are located on both sides of the capacitor; one is located at (x, y_1), where $y_1 > y_0$, and the other at

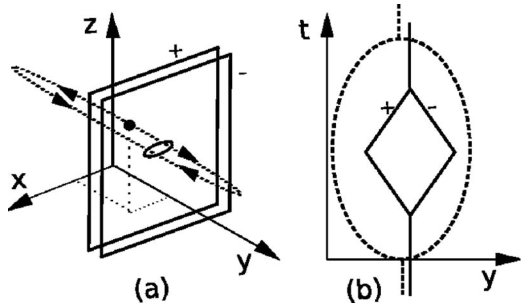

Fig. 1. A charged particle's wave packet (the dashed line) is split in the y direction. Each wave packet travels to a different side of a large capacitor, whose plates are zero distance apart. The two plates are moved apart and then together again in the y direction (the solid line). Afterward the wave packets interfere. Note that one of the wave packets travels back and forth through a little hole in the capacitor. (a) Schematic view of the system, (b) time dependence of the system.

(x,y_2), where $y_2<y_0$. Then at $t_0<t<t_1$ the wave packets interfere at (x,y). We shall see that the relative phase between the wave packets in this path is acquired from both the electric and magnetic effects.

Second, consider a particle moving along path 2 of Figs. 3 and 4, which we regard as the "magnetic path." At $t_0<t<t_1$ its wave packet is located at (x,y_0) where $x<x_0$. The wave packet then splits in the y direction and travels in the x direction. Next the wave packets interfere, so that they close a loop around one of the fluxons well within the time interval $[t_0,t_1]$. Again the wave packets travel through a little hole in the capacitor during which the distance between the plates vanishes. We shall see that the relative phase in this path is acquired solely from the magnetic effect.

We may ask what would be the relative phase acquired by wave packets that interfere without crossing the capacitor, as shown in path 3 of Fig. 5. It seems that the relative phase should be only an electric one, because the two magnetic phases acquired by each of the opposite fluxons would cancel each other. However, in this setup no relative phase is acquired at all as is explained in the following by choosing a specific gauge in which the four-vector potential vanishes along the path of the wave packets.

We continue by explicitly showing that the setup in Fig. 4 is non-radiating and deriving the relative phases. For convenience, we take $x_0=t_0=y_0=0$. We define the following singular vector potential:

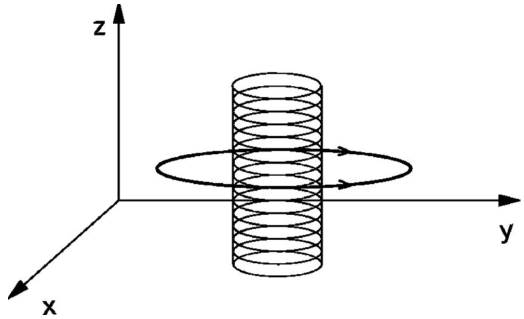

Fig. 2. A charged particle's wave packet interferes around a long solenoid.

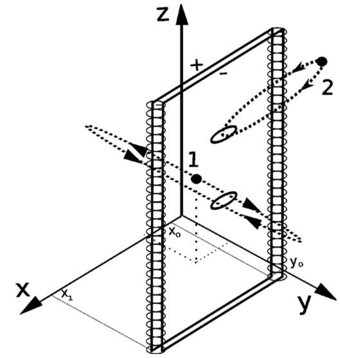

Fig. 3. Two capacitor plates are infinite in the z direction, finite in the x direction, and zero distance apart in the y direction. In addition, two infinitesimally thin and infinitely long solenoids are located on the edges of the capacitor. Paths 1 and 2 indicate two interfering wave packets of a charged particle. In path 1 (the electric path) the wave packet is split in the y direction. One of the wave packets travels back and forth through a little hole in the capacitor. In path 2 (the magnetic path) the wave packet is split in the y direction and travels in the x direction. Then the two wave packets interfere through a second hole in the capacitor.

$$\mathbf{A}=\pi[\theta(t)-\theta(t-T)][\theta(x)-\theta(x-L)]\delta(y)\hat{\mathbf{y}},\quad \phi=0, \tag{1}$$

where

$$\theta(x)=\begin{cases}0, & x<0\\ 1/2, & x=0\\ 1, & x>0\end{cases} \tag{2}$$

is the Heaviside step function. Although there are sources in the suggested setup, we choose to fix ϕ to zero, so that the gauge is fixed as the temporal gauge. $\mathbf{A}$ can be visualized as confined to the rectangle shown in Fig. 4 and pointing in the y direction. We have included a delta function $\delta(y)$ in Eq. (1) to make the potential singular to simplify the calculations; it is not an essential requirement.[2]

We then see that both the electric and magnetic fields vanish everywhere outside the source given by $\mathbf{A}$:

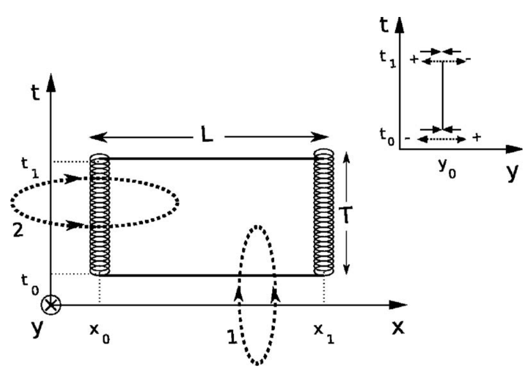

Fig. 4. At $t=t_0$ the two plates are instantaneously moved apart and then together again. Simultaneously, steady opposite currents are sent through the two solenoids beginning at $t=t_0$ and ending at $t=t_1$. At $t=t_1$ the plates are again instantaneously moved apart and then together, but in the opposite direction (as shown in the small graph). A charged particle, whose interfering wave packet is described by the electric path (path 1), acquires both an electric and magnetic phase. A charged particle whose wave packet encircles one of the fluxons, as described in the magnetic path (path 2), acquires only a magnetic phase.

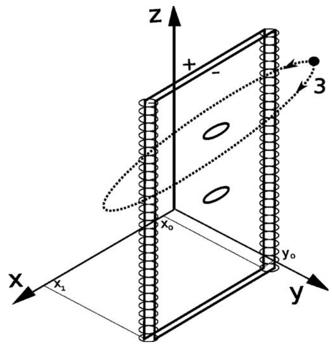

Fig. 5. Path 3: A charged particle wave packet interferes around the whole setup.

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}=-\pi[\delta(t)-\delta(t-T)][\theta(x)-\theta(x-L)]\delta(y)\hat{\mathbf{y}}, \tag{3a}$$

$$\mathbf{B}=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=\pi[\theta(t)-\theta(t-T)] \times[\delta(x)-\delta(x-L)]\delta(y)\hat{\mathbf{z}}. \tag{3b}$$

The electric field appears momentarily at $t=0$ and $t=T$ and the magnetic field is constant during $[0,T]$. The sources, that is, the charge and current densities that yield the vector potential (1) can be found using the differential form of Gauss' law and the potential form of Maxwell's equations (corresponding to the current gauge):

$$\nabla\cdot\mathbf{E}=4\pi\rho, \tag{4a}$$

$$\nabla^2\mathbf{A}-\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla(\nabla\cdot\mathbf{A})=-4\pi\mathbf{j}. \tag{4b}$$

The resulting charge density is

$$\rho=-\frac{1}{4}[\delta(t)-\delta(t-T)][\theta(x)-\theta(x-L)]\delta'(y). \tag{5}$$

We see that the charge density corresponds to capacitor plates that are instantaneously moved apart and then together again in the y direction at $t=0$ and $t=T$ (from the derivative of the delta function).

We separate the resulting current density $\mathbf{j}$ into two parts $\mathbf{j}=\mathbf{j}_c+\mathbf{j}_s$, where $\mathbf{j}_c$ corresponds to the capacitor plates' momentary movements at $t=0$ and $t=T$ and $\mathbf{j}_s$ corresponds to two infinitesimally thin solenoids that are turned on at $t=0$ and turned off at $t=T$, located on the edges of the capacitor:

$$\mathbf{j}_c=\frac{1}{4}[\delta'(t)-\delta'(t-T)][\theta(x)-\theta(x-L)]\ \delta(y))\hat{\mathbf{y}}, \tag{6a}$$

$$\mathbf{j}_s=\frac{1}{4}[\theta(t)-\theta(t-T)]\nabla\times\{[\delta(x)-\delta(x-L)]\delta(y)\hat{\mathbf{z}}\}. \tag{6b}$$

We can verify that the continuity equation,

$$\nabla\cdot\mathbf{j}+\partial\rho/\partial t=0, \tag{7}$$

holds.

We now separate the acquired phase into the electric and magnetic phases. As Eq. (1) shows, the relative phase acquired when a charged particle's wave packet crosses $y=0$ is

$$\theta=\oint A^\mu dx_\mu=\pi[\theta(t)-\theta(t-T)][\theta(x)-\theta(x-L)], \tag{8}$$

which equals π if the particle crosses the topology in Fig. 4. This phase may originate from the electric effect induced by the charges as well as the magnetic one induced by the solenoidal currents.

To obtain the electric and magnetic parts of the phase we transform to the Coulomb gauge.[3] In this gauge the scalar potential depends only on the charge density and the vector potential depends only on the transverse current density $\mathbf{j}_t$ for which $\nabla\cdot\mathbf{j}_t=0$:[4]

$$\nabla^2\phi=-4\pi\rho, \tag{9a}$$

$$\nabla^2\mathbf{A}-\partial^2\mathbf{A}/\partial t^2=-4\pi\mathbf{j}_t. \tag{9b}$$

Therefore the electric and magnetic phases are clearly distinct in the Coulomb gauge.

The gauge transformation of the four-vector potential

$$\mathbf{A}\to\tilde{\mathbf{A}}=\mathbf{A}+\nabla\Lambda, \tag{10a}$$

$$\phi\to\tilde{\phi}=-\partial\Lambda/\partial t \tag{10b}$$

to the Coulomb gauge, where $\nabla\cdot\tilde{\mathbf{A}}=0$, requires solving the Poisson equation, $\nabla^2\Lambda=-4\pi C$, in two dimensions (because the setup is invariant under replacements along the z axis), where

$$C=\frac{1}{4\pi}\nabla\cdot\mathbf{A}=\frac{1}{4}[\theta(t)-\theta(t-T)][\theta(x)-\theta(x-L)]\delta'(y). \tag{11}$$

The Green's function of the Poisson's equation in two dimensions, $\nabla^2\Lambda=4\pi\delta(x-x_0)\delta(y-y_0)$, is[5]

$$G(x,y|x_0,y_0)=\mathrm{Re}[-2\ln(\omega-\omega_0)], \tag{12}$$

where $\omega\equiv x+iy$. Therefore,

$$\Lambda=\int C(x',y',t')\ln[(x-x')^2+(y-y')^2]dx'dy'. \tag{13}$$

If we integrate twice by parts and cancel the boundary terms, we obtain

$$\Lambda=[\theta(t)-\theta(t-T)]F(x,y), \tag{14}$$

where

$$F(x,y)=\frac{1}{2}\left[\arctan\left(\frac{x}{y}\right)-\arctan\left(\frac{x-L}{y}\right)\right]. \tag{15}$$

The corresponding potentials are then:

$$\tilde{\phi}=-\frac{1}{2}[\delta(t)-\delta(t-T)]\left[\arctan\left(\frac{x}{y}\right)-\arctan\left(\frac{x-L}{y}\right)\right], \tag{16a}$$

$$\tilde{A}_x=\frac{1}{2}[\theta(t)-\theta(t-T)]\left[\frac{y}{x^2+y^2}-\frac{y}{(x-L)^2+y^2}\right], \tag{16b}$$

$$\widetilde{A}_y = -\frac{1}{2}[\theta(t) - \theta(t-T)]\left[\frac{x}{x^2+y^2} - \frac{x-L}{(x-L)^2+y^2}\right]. \tag{16c}$$

Note that $\widetilde{A}_y$ is now non-singular because the singular term A_y cancels, as can be verified from Eq. (17a). In addition, we can verify Eq. (9) by taking into account that

$$\lim_{x\to 0} \frac{\partial}{\partial y} \arctan\left(\frac{x}{y}\right) = -\pi[\theta(x) - \theta(-x)]\delta(y), \tag{17a}$$

$$\lim_{x\to 0} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = 2\pi\delta(x)\delta(y), \tag{17b}$$

$$\lim_{x\to 0} \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = 0, \tag{17c}$$

where we have used $\theta(x)+\theta(-x)=1$ and the definition

$$\lim_{x\to 0} \frac{1}{\pi}\frac{x}{x^2+y^2} = [\theta(x) - \theta(-x)]\delta(y). \tag{18}$$

To show the path dependence of the electric and magnetic phases, we first discuss wave packets that interfere in the electric path, corresponding to path 1 of Fig. 4. If the charged particle's wave packets are located at (x,d) and $(x,-d)$ at $t=0$ and interfere before $t=T$, the resultant electric and magnetic phases are

$$\theta_e = \oint \tilde{\phi} dt = -[F(x,-d) - F(x,d)] = \arctan\left(\frac{x}{d}\right) - \arctan\left(\frac{x-L}{d}\right), \tag{19a}$$

$$\theta_m = \oint \widetilde{\mathbf{A}} \cdot d\mathbf{l} = \int_d^{-d}\left(\frac{\partial F(x,y)}{\partial y} + A_y\right)dy = -\theta_e + \pi[\theta(x) - \theta(x-L)]. \tag{19b}$$

The sum of the electric and the magnetic phases equals π if $0<x<L$ and vanishes otherwise.

The function $F(x,y)$ in Eq. (15) is shown in Fig. 6 as a function of y for various values of x at $t=0$. From Eq. (19) it can be immediately seen that if the distance $2d$ between the wave packets in the y direction at $t=0$ is small compared to L (that is, $0<d\ll L$ and $0<x<L$), the electric phase equals π and thus the magnetic one vanishes. As d increases, the electric phase becomes less dominant, and the magnetic phase becomes more dominant because the sum of the two phases still equals π. The decrease in the electric phase can be explained by the fact that the capacitor is now finite. Note that if $x(t=0)<0$ and $x(t=0)>L$, or if both wave packets are located on the same side of the capacitor at $t=0$, then the electric and magnetic phases cancel. Furthermore, if the wave packets interfere only after $t=T$, the total phase is zero.

We next examine wave packets that interfere in the magnetic path corresponding to path 2 of Fig. 4. It can easily be checked that the wave packets that interfere around (0,0) or $(L,0)$ (not both) well within $[0,T]$ acquire only a magnetic phase of π. From the definition of the vector potential (1), it can be seen that no relative phase is acquired in path 3 (see Fig. 5). Here again the magnetic and electric phases cancel.

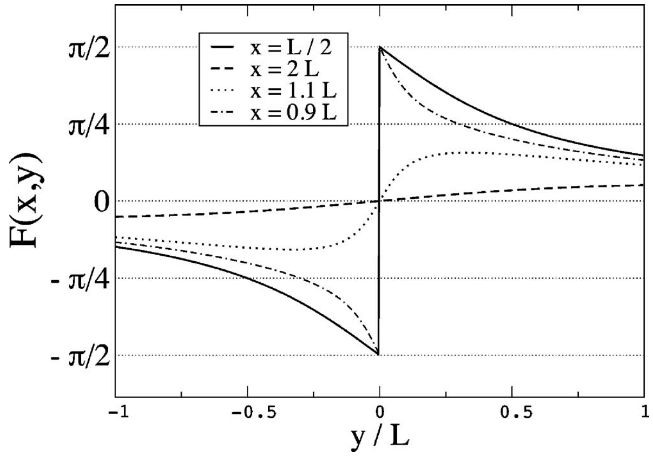

Fig. 6. Spatial dependence of $\tilde{\phi}(x,y,t)$, where the scalar potential in the Coulomb gauge is given as a function of y for various values of x. If the particle's wave function consists of two wave packets that are located at (x,y_1) and (x,y_2) at $t=0$ and interfere before $t=T$, then the electric phase is given by $F(x,y_1)-F(x,y_2)$ (for $y_1>y_2$). For $0<x<L$, $F(x,y)$ has a singularity at $y=0$, for which the sum of the electric and magnetic phases is non-trivial.

We now generalize the previous discussion to any singular gauge. This generalization manifests itself as a symmetry between the electric and magnetic Aharonov–Bohm effects. For simplicity, let space be two dimensional. In the previous method the solenoids in Fig. 4 represent two opposite magnetic dipoles at rest and the horizontal lines represent two wires of the capacitor that are quickly opened and closed. Suppose that initially the opposite magnetic dipoles are close together and are boosted in opposite directions along the x axis for a time T. Then each turns around and travels with the same velocity v so that they meet after a second time T. The corresponding potential is

$$\mathbf{A} = \pi\theta(vt-x)\theta(vt+x)\theta[-v(t-T)+x] \times \theta[-v(t-T)-x]\delta(y)\hat{\mathbf{y}}, \tag{20a}$$

$$\phi = 0, \tag{20b}$$

which corresponds to the rhombus in Fig. 7. If $v=0$, the potential reduces to just two opposite magnetic dipoles at rest at $x=0$. For $0<v<1$, we obtain the time-like setup,

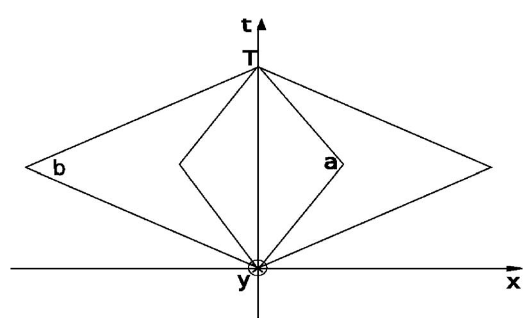

Fig. 7. (a) In the time-like setup the vector potential points in the y direction and is confined inside the rhombus. The vector potential corresponds to two opposite magnetic dipoles that travel through its boundaries and two electric dipoles that travel on the same track, where the electric part is weaker. (b) Space-like setup, describing the same scheme but where the electric dipoles dominate.

which is visualized in Fig. 7(a). Because the magnetic dipoles are boosted, the densities correspond to both magnetic and electric dipoles. By using the same considerations as in Eq. (4), we obtain charge and current densities that correspond to two opposite magnetic dipoles and two electric dipoles that travel on the boundary of the vector potential. The magnetic dipoles travel in opposite directions along the x axis. The electric dipoles start traveling in opposite directions along the x axis with a positive sign during the interval $[0,T]$ and then are flipped to a negative sign during $[T,2T]$. Because $v<1$, the magnetic part is more dominant than the electric one:

$$\rho = -\frac{v}{2}[f_+(t,x) - f_-(t,x)]\delta'(y), \tag{21a}$$

$$\mathbf{j}_s = \frac{1}{2}\boldsymbol{\nabla} \times [g_+(t,x) - g_-(t,x)]\delta(y)\hat{\mathbf{z}}, \tag{21b}$$

where $f_\pm(t,x)$ define the paths for the electric dipoles, and $g_\pm(t,x)$ define the paths for the magnetic dipoles:

$$f_+(t,x) = [\delta(vt-x) + \delta(vt+x)][\theta(t) - \theta(t-T/2)] \times [\theta(x+vT/2) - \theta(x-vT/2)], \tag{22a}$$

$$f_-(t,x) = \{\delta[-(v(t-T)-x\ ] + \delta[-v(t-T)+x]\} \times [\theta(t-T/2) - \theta(t-T)][\theta(x+vT/2) - \theta(x-vT/2)], \tag{22b}$$

$$g_+(t,x) = [\delta(vt-x) + \delta(-v(t-T)-x)] \times [\theta(t) - \theta(t-T)][\theta(x) - \theta(x-vT/2)], \tag{22c}$$

$$g_-(t,x) = [\delta(vt+x) + \delta(-v(t-T)+x)] \times [\theta(t) - \theta(t-T)][\theta(x+vT/2) - \theta(x)]. \tag{22d}$$

Is there also a meaning to a space-like setup as illustrated in Fig. 7(b) with $v>1$? We can illustrate the two horizontal lines in Fig. 4 as two electric dipoles located at the center of the wire at $t=0$ that travel with infinite velocity in opposite directions. (Take $v\to\infty$ in Eq. (20). No real superluminal velocity is assumed.) In such a system no magnetic dipoles appear. However, if we "boost" the electric dipoles with a finite velocity greater than 1, we obtain a potential corresponding to Fig. 7(b), resulting in both electric and magnetic dipoles, with now the electric part dominant, as can be seen from Eq. (21).

We note that the combination of the Aharonov–Bohm effects can also be achieved using finite capacitors, specifically circular capacitors, as illustrated in Fig. 8. The configuration is similar to Eq. (1). We instantaneously move the plates apart and then together again. At the same time a constant current $\mathbf{j}_s$ flows through an infinitesimally thin toroidal solenoid that encircles the capacitor just at the boundaries of the capacitor. The current loops around the torus. After a time ΔT we instantaneously move the capacitor plates apart and then together again, but in the opposite directions, and simultaneously send an opposite current through the toroidal solenoid, so that the currents cancel each other. In analogy to Eq. (1), we choose

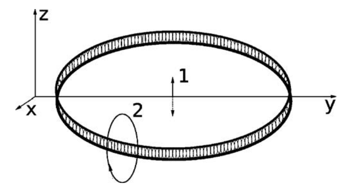

Fig. 8. The current flows inside an infinitesimal toroidal solenoid during the interval $[0,T]$. The capacitor's plates are simultaneously moved apart and then together again at $t=0$ and at $t=T$ (in opposite directions). A charged particle's wave packet that travels in paths 1 or 2 can acquire both electric and magnetic phases.

$$\mathbf{A} = \pi[\theta(t) - \theta(t-T)]\theta(L-r)\delta(z)\hat{\mathbf{z}}, \quad \phi = 0, \tag{23}$$

expressed in cylindrical coordinates. From Eq. (4) we obtain the predicted charge and current densities:

$$\rho = -\frac{1}{4}[\delta(t) - \delta(t-T)]\theta(L-r)\delta'(z), \tag{24a}$$

$$\mathbf{j}_s = \frac{1}{4}[\theta(t) - \theta(t-T)]\boldsymbol{\nabla} \times [\delta(L-r)\delta(z)\hat{\phi}]. \tag{24b}$$

The electric and magnetic phases are extracted by transforming the vector potential (23) to the Coulomb gauge. A somewhat similar non-radiating setup has also been proposed by Afanasiev,[6] where the current was taken to be linearly dependent on time and the capacitor was static. Such a setup was proposed there as a time-dependent Aharonov–Bohm effect. In this respect, the proposed scheme may suggest nonclassical information transfer applications using time-dependent Aharonov–Bohm effects. Although in the original magnetic Aharonov–Bohm effect only one topological bit of information is encoded in the relative phase, in the time-dependent setup many bits of information can be encoded in the topology without sending any classical traces such as electromagnetic fields into the rest of the world.

In conclusion, we emphasize that gauge invariance is manifested in quantum mechanics by the acquirement of a relative phase when the wave function passes through a non-simply-connected region. The electric and magnetic Aharonov–Bohm effects are distinctively manifested in the Coulomb gauge. In any configuration, the total relative phase, electric plus magnetic, is gauge invariant.

The authors thank J. Kupferman, S. Popescu, A. Casher and M. Marcovitch for helpful discussions. This work has been supported by the Israel Science Foundation Grant No. 784/06.

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[1]Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in the quantum theory," Phys. Rev. **115**, 485–491 (1959).

[2]The same effect can be obtained by choosing a smooth non-singular vector potential $\mathbf{A}(\mathbf{x},t)$ that is confined in both space and time. Outside the confined region the potential vanishes. Within a certain part of the confined region, it is constant. In the intermediate area it smoothly decays from a constant value to zero. It is only in this region that the electric and magnetic fields do not vanish. Such a potential imposes charge and current densities different from Eqs. (5) and (6), but the setup is still non-radiating. The same combined electric and magnetic Aharonov–Bohm effects are obtained if the two interfering wave packets do not enter (in

space-time) the intermediate regions, where **E** and **B** are non-trivial. We have used the delta (and step) function(s) to simplify the calculations, specifically those of the charge and current densities.

[3]When treating the sources quantum mechanically, the Coulomb gauge is essential because it describes the back reaction correctly. See Y. Aharonov and J. Anandan, "Is there a preferred canonical quantum gauge," Phys. Lett. A **160**, 493–497 (1991).

[4]Note that $\mathbf{j}=\mathbf{j}_l+\mathbf{j}_t$, where $\mathbf{j}_l$ is the longitudinal current for which $\nabla\times\mathbf{j}_l=0$. This standard separation of **j** does not coincide with $\mathbf{j}_c$ and $\mathbf{j}_s$ in Eq. (6) as can be verified from Eqs. (9) and (16).

[5]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw–Hill, New-York, 1953), p. 1175.

[6]G. N. Afanasiev and Yu P. Stepanovsky, "The Electromagnetic field of elementary time-dependent toroidal sources," J. Phys. A **28**, 4565–4580 (1995); G. N. Afanasiev, *Topological Effects in Quantum Mechanics* (Kluwer, Dordrecht, 1999), pp. 205–206.

Medical Machine. Early treatment with electricity used shocks delivered by Leiden jars, but after the discovery of electromagnetic induction by Michael Faraday in 1831 it was possible to deliver shocks using Magneto-Electric Machines. Bobbins wound with many turns of fine wire revolved in the magnetic field of a U-magnet. The patient grasped the electrodes, and took the shock to relieve all sorts of ills: paralysis, palsies, rheumatism, tumors, sprains, chilblains, inflammations, incontinence. Presumably the placebo effect caused a certain number of cures. This machine I found in an antique shop in Lambertville, New Jersey. It was manufactured by W. H. Burnap of New York, and his signature can be seen across the central block of print. On the right-hand side of the label is a testimonial by Charles Grafton Page from 1854, and on the left-hand side is a testimonial from Benjamin Silliman, Snr. of Yale University. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)