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On the Measurement of Velocity of Relativistic Particles.

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1. – Introduction.

The question of the measurement of the velocity for a particle satisfying Dirac's equation has recently been raised by KOBA [1]. In this paper, we wish to subject this question to additional discussion with the purpose of further clarifying the problem.

The question under discussion arose from a suggestion of DIRAC [2] that the eigen values, $\pm C$, for the component of the velocity in any direction, can be understood by means of the following measurement. At the time $t = 0$ the particle is allowed to pass through a definite position, say $z = 0$. Its wave function at this time is a very narrow packet centered at $z = 0$ (we assume that P_x and P_y equal zero). Such a wave packet implies, when $\Delta z \to 0$, very high Fourier components, so that in momentum space the major statistical weight is in the region where $|P| > mc$ and where $|v|$ is therefore very close to C. DIRAC proposed this measurement as a way of explaining the fact that the only possible eigen values of any component of velocity for a particle satisfying the Dirac equation are $\pm C$.

(1) Z. KOBA: *Nuovo Cimento*, **3**, 1 (1956).
(2) P. A. M. DIRAC: *Principles of Quantum Mechanics* (Oxford, 1947), p. 261.

KOBA [1] has criticized the above conclusion of DIRAC and we shall analyse both DIRAC's proposed measurement and KOBA's criticism in the present paper. In particular we shall show that the above measurement does not demonstrate the essential difference between the properties of the velocity of particles that satisfy the Dirac equation and of those for which we simply assume the above relativistic relation between energy E and momentum p with the requirement that the velocity be equal to $\partial E/\partial \varrho$. We shall then point out further measurements which would be able to show up the essential difference between the motions of these two types of particles.

2. – Measurements of velocity for relativistic particles not satisfying Dirac's equation.

In order to bring out clearly the characteristic new properties of the velocity for a particle satisfying the Dirac equation, we shall first discuss the motion of a spinless particle, which can have only positive energies, and for which [3] the energy is therefore given by

$$E = c\sqrt{P^2 + m^2c^2}\,. \tag{1}$$

(We restrict ourselves to positive energies because as we shall see later, the essential new properties of the velocity in the Dirac equation are inextricably bound up with the possibility of negative energies).

To treat this problem, we start with a wave packet for which $P_x = P_y = 0$, and which at $t = 0$ is given by $\delta(z)$. We then obtain for the time dependent wave functions

$$\psi = \lim_{k_m \to \infty} \int_{-k_m}^{k_m} \exp\left[i(kx - \sqrt{k_0^2 + k^2}\, ct)\right] \mathrm{d}k\,. \tag{2}$$

where

$$K_0 = \frac{mc}{\hbar}\,, \tag{3}$$

and K_m is of the order of $\Delta k = \Delta P/\hbar$, where ΔP is the spread of the wave packet in momentum space.

It is clear that as $K_m \to \infty$, the major part of the probability in momentum space corresponds to $|k| \gg k_0$, and therefore, to $|v|$ close to C. For,

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(3) See P. A. M. DIRAC: *Quantum Mechanics* (Oxford, 1947) (third edition), chap. XI, equ. (3). Here DIRAC begins by considering relativistic wave equation without spin for positive energies only.

in order to normalize the wave function (2), one should divide it by $\sqrt{2K_m}$. The part of the integral between $-K_0$ and $+K_0$ is then equal to $\sqrt{2K_0}$: Thus, as $K_m \to \infty$, the probability that $|K| < K_0$ approaches zero; and indeed, the same is true for $|K| < K_1$, where K_1 is any predetermined finite number.

In order to estimate the integral (2) for large values of K_m, let us therefose choose $K_1 \gg K_0$, but $K_1 \ll K_m$.

We can then use the approximation

$$\sqrt{K_0^2 + k^2} \cong |K| + \left|\frac{K_0^2}{2K}\right|, \tag{4}$$

provided that we carry out the integration from $-\infty$ to $-K_1$ and from K_1 to ∞. We also expand

$$\exp\left[i\,\frac{k_0^2 ct}{2k}\right] \cong \left(1 + i\,\frac{k_0^2 ct}{2k}\right),$$

provided that we choose times such that

$$k_0^2 ct/2k_1 \ll 1\,. \tag{5}$$

If ΔZ is chosen small enough (so that $K_m \cong (1/\Delta z)$ is large) then k_1 can be chosen big enough so that t can be arbitrarily large.

We then obtain;

$$\psi(z,t) \cong \int\limits_{-k_m}^{-k_1} \exp\left[ik(z+ct)\right]\left(1 + \frac{ik_0^2 ct}{2k}\right) dk + \int\limits_{k_1}^{k_m} \exp\left[ik(z-ct)\right]\left(1 - i\,\frac{ik_0^2 ct}{2k}\right) dk\,.$$

We begin by considering the major terms (i.e., those not involving $ck_0^2 t/2k$). These give

$$\psi \cong \frac{2\exp\left[-i\left(\dfrac{k_1+k_m}{2}\right)(z+ct)\right]\sin\dfrac{(k_m-k_1)}{2}(z+ct)}{z+ct} + \\ + \frac{2\exp\left[i\,\dfrac{(k_1+k_m)}{2}(z-ct)\right]\sin\dfrac{(k_m-k_1)}{2}(z-ct)}{z-ct}\,.$$

As K_1 and K_m approach ∞ the first term in the above equation represents $\delta(Z+ct)$ multiplied by a phase factor $\exp[-i(k_1+k_m/2)(Z+ct)]$ while the second term represents $\delta(Z-ct)$ multiplied by a phase factor

$\exp[i(k_1+k_m/2)(Z-ct)]$. The phase factor describes the mean momenta of the packets. Thus, the original δ-function splits into two parts, moving in opposite directions with the velocity of light. Each packet has a high momentum.

A simple calculation shows that the terms involving $CK_0^2t/2K$ lead to finite integrals, having a negligible probability as K_1 and K_m approach infinity. (This is because the probability is proportional to $1/K^2$, which leads to a convergent integral, while the cross products with the major term are of no interest here, because they represent only the interference with the δ-function).

We can explain the above result by noting that practically all the particles have a velocity $\dot{v}_z = \pm c$. Thus, the major part of the wave function must split into two very narrow packets with corresponding velocities. The spread of the wave packet due to the particles with $|k| < k_1$ can be neglected, because they constitute a negligible part of the total number of particles. However, the velocity is given by

$$v = \frac{ck}{\sqrt{k_0^2+k^2}} \simeq c\left(1-\frac{k_0^2}{2k^2}\right). \tag{7}$$

In the time, t, this will produce a spread $\Delta Z = \Delta v \cdot t$, which is of the order of $K_0^2ct/2K_1^2$.

If we satisfy the criterion (5), we then obtain

$$\Delta Z \cdot k_1 \ll 1 \qquad \text{or} \qquad \Delta Z \ll \frac{1}{k_1}. \tag{8}$$

But $1/k_1$ is a quantity much less than λc, the Compton wave length. Thus, we can be assured that the spread of the wave packet within the time, t, is quite negligible. But as we have seen from equ. (7), the indeterminacy in the velocity is independent of time, and depends only on k_0/k_1.

We now return to a discussion of DIRAC's proposed measurement of the velocity for the case of a particle that does not satisfy Dirac's equation. To measure this velocity, we could then measure the position of the particle after waiting for a time, t, and the velocity would be given by $v = z/t$. It is evident that if k_1 is chosen very large (i.e., a very narrow initial wave packet), then the values of the velocity obtained in this measurement will be $\pm C$. Moreover, if we allow t to be large, we can make a comparatively rough measurement of the position at the time t. The change in the momentum ($\Delta P \simeq h/\Delta Z$) will then be small compared with the mean momentum ($|\overline{P}| \gg \hbar k_1$), so that the change in velocity will then be negligible. This means that in such a measurement the electron is left with an eigenvalue of the velocity.

3. – Motion of particles satisfying Dirac's equation.

Before considering the experiment proposed by DIRAC for the case of particles satisfying the Dirac equation, we shall summarize some of the new properties of the motion, implied by the Dirac equation.

The most essential new characteristic of the motion is the independence of the velocity operator $\boldsymbol{\alpha}$ and the momentum operator, $\boldsymbol{p}$. (This also makes possible the commutation of $\boldsymbol{\alpha}$ and $\boldsymbol{x}$). One of the most important manifestations of the independence of $\boldsymbol{\alpha}$ and $\boldsymbol{p}$ are the well known « Zitterbewegungen » which constitute a fluctuation motion of the velocity around an average equal to

$$V = \frac{pc}{\sqrt{p^2 + m^2c^2}}\,.$$

In addition, the Dirac velocity operator, $c\alpha_z$, has only two eigenvalues, $\pm C$, while in the non-Dirac case, that we have discussed in Sect. **2**, the velocity can take any value between $+c$ and $-c$.

We shall in this article use the representation in which $\gamma^5 = \gamma^1\cdot\gamma^2\cdot\gamma^3\cdot\gamma^4$ is diagonal. We then have

$$\text{(9)}\qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boldsymbol{\alpha} = \gamma^5\boldsymbol{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\boldsymbol{\sigma} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\boldsymbol{\sigma}$ is now the two row matrix of the Pauli theory.

Hereafter, we shall restrict ourselves to eigenstates of σ_z, with $\sigma_z = 1$. We also consider only states with $Px = Py = 0$. (These states are general enough to illustrate all of our essential ideas). We then need consider only two components for our spinor wave function, which we label ψ_1, and ψ_2 respectively. Dirac's equation then becomes

$$\text{(10)}\qquad (E - CP)\psi_1 = mc^2\psi_2\,, \qquad (E + CP)\psi_2 = mc^2\psi_1\,.$$

The normalized eigenfunctions corresponding to definite energy and momentum are (for positive and negative energies respectively)

$$\text{(11)}\qquad \psi_+ = \begin{pmatrix} 1 \\ a \end{pmatrix}\frac{1}{\sqrt{1 + a^2}}\,, \qquad \psi_- = \begin{pmatrix} 1 \\ b \end{pmatrix}\frac{1}{\sqrt{1 + b^2}}\,,$$

with:

$$\text{(12)}\qquad a = \frac{mc^2}{|E| + pc}\,, \qquad b = \frac{mc^2}{-|E| + pc}\,.$$

Let us now consider the case where $p = 0$. In this case, we obtain

$$\psi_+ = \begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}, \qquad \psi_- = \begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}. \tag{13}$$

The eigenstates of α_2 are, in this representation

$$\varphi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{14}$$

Thus, we have

$$\psi_+ = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \qquad \psi_- = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2). \tag{15}$$

An arbitrary initial wave function can be written as

$$\psi = a\varphi_1 + b\varphi_2 = \frac{a+b}{\sqrt{2}}\psi_+ + \frac{a-b}{\sqrt{2}}\psi_-. \tag{16}$$

With the passage of time, we then obtain

$$\psi = \frac{a+b}{\sqrt{2}}\psi_+ \exp\left[\frac{-im_0c^2t}{\hbar}\right] + \frac{a-b}{\sqrt{2}}\psi_- \exp\left[\frac{im_0c^2t}{\hbar}\right]. \tag{17}$$

Let us choose a state for which $v = c$ at $t = 0$ (so that $b = 0$). This gives

$$\psi = \sqrt{2}\,\varphi_1 \cos m_0c^2t/\hbar + \sqrt{2}\,\varphi_2 \sin m_0c^2t/\hbar. \tag{18}$$

The mean velocity then oscillates between $+c$ and $-c$ with the Compton frequency, $\omega_c = \hbar/M_0C^2$. These oscillations are precisely the Zitterbewegungen.

Let us now make a Lorentz transformation on wave function (16), by a velocity, in the z direction, of $\omega = c \operatorname{tg} h\theta$. As is well known ([4]) under such a transformation, the wave function is multiplied by the matrix, $M = \exp[\alpha z(\theta/2)]$. This gives

$$\varphi_1 \to \exp\left[\frac{\theta}{2}\right]\varphi_1, \qquad \varphi_2 \to \exp\left[-\frac{\theta}{2}\right]\varphi_2 \tag{19}$$

and

$$\psi \to a \exp\left[\frac{\theta}{2}\right]\varphi_1 + b \exp\left[-\frac{\theta}{2}\right]\varphi_2. \tag{20}$$

([4]) See P. A. M. DIRAC: *Quantum Mechanics*, Chap. XI (Reference ([3])).

Hence, as $\theta \to \infty$, ψ will contain mainly φ_1.

We now consider the Lorentz transformation of the specific wave function (18). Here, we must also write

$$t = \left(t' - \frac{\omega z}{c^2}\right)\frac{1}{\sqrt{1-(\omega^2/c^2)}} = \frac{E't' - p'z'}{m_0 c^2},$$

where

$$E' = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \qquad \text{and} \qquad p' = \frac{m_0 \omega}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{\omega}{c}.$$

We get

$$\psi \to \sqrt{2}\exp\left[\frac{\theta}{2}\right]\varphi_1 \cos\left(\frac{(p'z' - E't')}{\hbar}\right) - \sqrt{2}\, i \exp\left[-\frac{\theta}{2}\right]\varphi_2 \sin\left(\frac{(p'z' - E't')}{\hbar}\right).$$

As $\omega \to c$, we neglect the latter term, and obtain

$$\psi \to \frac{1}{\sqrt{2}}\varphi_1 \exp\left[\frac{i}{\hbar}(p'z' - E't')\right] + \frac{1}{\sqrt{2}}\varphi_1 \exp\left[\frac{i}{\hbar}(-p'z' + E't')\right]. \tag{21}$$

The main conclusion is that as $\omega \to c$, $|p'| \to \infty$, but positive energy appears with positive momentum and negative energy with negative momentum. The term involving φ_2 becomes very small. The Zitterbewegungen must therefore become negligible. This may be understood as a result of the Lorentz transformation which multiplies the amplitude of the Zitterbewegungen in the frame in which $p = 0$ by a factor of $\sqrt{1-\beta^2}$. Moreover the velocity becomes well defined even though the momentum fluctuates between $+\infty$ and $-\infty$. This illustrates the independence of the velocity and the momentum in the Dirac equation.

We can also understand the reduction of the Zitterbewegungen at high momentum with the aid of a formula given by DIRAC ([5]) for the behavior of the operator α_z as a function of the time

$$\alpha_z = \frac{i\hbar}{2}\dot{\alpha}_{z0} \exp\left[\frac{-2iHt}{\hbar}\right] H^{-1} + Cp_z H^{-1}, \tag{22}$$

where $H = C\boldsymbol{\alpha}_1 \boldsymbol{p} + \beta mc^2$ and $\dot{\alpha}_{z0}$ is the initial value of $\dot{\alpha}_z$.

The first term in (22) represents an oscillation of α_z around a mean defined by $P_z/H = P/\sqrt{P^2 + m^2c^2}$, which is just what we would obtain in the non-Dirac case treated in Sect. 2. This term represents the Zitterbewegungen.

([5]) P. A. M. DIRAC: *The Principles of Quantum Mechanics* (Oxford, 1947), third edition, p. 262, equ. (28).

As $|H| = |E|$ approaches infinity, however, it is clear that the first term in (22) becomes negligible in comparison to the second. Thus, we conclude once again that the Zitterbewegungen become negligible ([6]).

The essential conclusion of this section is then that for a Dirac particle of infinite momentum, there are no Zitterbewegungen, while for a particle of very high momentum, the latter become negligible. Hence, the special characteristics of the motion in the Dirac theory become less and less significant as the value of $|P|$ is increased without limit.

4. – Measurement of velocity for particles satisfying the Dirac equation.

We are now ready to discuss the measurement of velocity suggested by Dirac for the case of particles satisfying the Dirac equation. As in the case considered in Sect. 2, the first step is to allow the particle to pass through a definite position (say $z = 0$) at $t = 0$ with a very small error, Δz, so that the wave function becomes approximately $\delta(z)$ (Note that $P_x = P_y = 0$ and that $\sigma_z = 1$). We now express an arbitrary wave function of this kind as a superposition of eigenfunctions of E and P. To compare with Koba's result, we consider a case in which $\overline{P} = 0$. Since $\boldsymbol{x}$ and $\boldsymbol{\alpha}$ commute, we can choose a δ-function that is an arbitrary linear combination of eigenfunction of α_z. Thus,

$$\psi_{t=0} = A \int_{-\infty}^{\infty} \varphi_1 \exp[ikz]\, \mathrm{d}k + B \int_{-\infty}^{\infty} \varphi_2 \exp[ikz]\, \mathrm{d}k\,,$$

where A and B are arbitrary constants.

In order to solve for $\psi(z, t)$, we express φ_1 and φ_2 in terms of ψ_+ and ψ_-. By equ. (11), we obtain

$$(23) \qquad \left\{ \begin{aligned} \varphi_1 &= \frac{b\sqrt{1+a^2}\,\psi_+ - a\sqrt{1+b^2}\,\psi_-}{b-a}\,, \\ \varphi_2 &= \frac{\sqrt{1+a^2}\,\psi_+ - \sqrt{1+b^2}\,\psi_-}{a-b}\,. \end{aligned} \right.$$

([6]) The frequency of this motion becomes infinite. At first sight one would suppose that this too should approach zero because the rate of a moving clock is reduced by the factor $\sqrt{1-\beta^2}$. But this reduction occurs only if we consider a point moving with velocity v; whereas in the evolution of the wave function, we consider a point fixed in the laboratory system of co-ordinates; a simple calculation shows that for this case, the frequency increases in the ratio of $\sqrt{1-\beta^2}$.

We note that ψ_+ oscillates with the factor $\exp[-i\sqrt{k_0^2+h^2}\,ct]$ and ψ_- with the factor $\exp[i\sqrt{k_0^2+h^2}\,ct]$. As we did in Sect. 2, we can for large k expand $\sqrt{k_0^2+k^2} \cong k(1+k_0^2/2k)$, and neglect the second term in the expansion, under the same limitations (5) that held for the non-Dirac case. For in both cases, the major part of the integral will come from very large values of k. We then obtain

$$\psi(z,t) \cong \int_{k_1}^{k_m} \mathrm{d}k \left[\exp[ik(z-ct)]\frac{Ab-B}{b-a}\sqrt{1+a^2}\,\psi_+ + \right.$$
$$\left. + \exp[ik(z+ct)]\frac{B-aA}{b-a}\sqrt{1+b^2}\,\psi_-\right] + \int_{-k_m}^{-k_1}\mathrm{d}k\left[\exp[ik(z+ct)]\cdot\right.$$
$$\left.\cdot\frac{Ab-B}{b-a}\sqrt{1+a^2}\,\psi_+ + \exp[ik(z-ct)]\frac{B-aA}{b-a}\sqrt{1+b^2}\,\psi_-\right].$$

Then by equ. (12), we see that as $k\to+\infty$,

$$\frac{Ab-B}{b-a}\sqrt{1+a^2}\,\psi_+ \to A\varphi_1\,, \qquad \frac{B-aA}{b-a}\sqrt{1+b^2}\,\psi_- \to B\varphi_2\,,$$

while as $k\to-\infty$

$$\frac{Ab-B}{b-a}\sqrt{1+a^2}\,\psi_+ \to B\varphi_2\,, \qquad \frac{B-aA}{b-a}\sqrt{1+b^2}\,\psi_- \to A\varphi_1\,.$$

We make the above approximations when $|k|>k_1$, and then we may, with negligible error extend the integration to include the region where $|k|<k_1$ (because, as pointed, out in Sect. 2, this region has negligible probability). We then obtain

$$\psi(z,t) \cong \int_{-k_m}^{k_m} A\varphi_1 \exp[ik(z-ct)]\,\mathrm{d}k + \int_{-k_m}^{k_m} B\varphi_2 \exp[ik(z+ct)]\,\mathrm{d}k\,, \tag{24}$$

as $k_m\to\infty$, this becomes

$$\psi(z,t) \cong A\,\delta(z-ct)\varphi_1 + B\,\delta(z+ct)\varphi_2\,. \tag{25}$$

We see from the above equation that the wave function splits into two packets, which are eigenfunctions of α_z corresponding to $\alpha_z=+1$ and to $\alpha_z=-1$. These packets move with velocity $+c$ and $-c$ respectively. In agreement with Koba, we obtain the result that for a given velocity, the momentum may either be positive or negative, but that the sign of the momentum and that of the energy are coupled. The wave function appearing in equ. (24) also agrees with that obtained in a special case (with $A=B$) by Lorentz transformation of the wave function of zero momentum (see equ. (21)).

We conclude from equ. (25) that the behaviour of Dirac particles after they have been localized very precisely does not differ essentially from the behavior of the non-Dirac particles discussed in Sect. **2**. Moreover, we see also that at least for times shorter than those in the criterion (5), the wave function does not spread significantly.

In addition, the particle velocity does not fluctuate significantly. Evidently, as in the case discussed in Sect. **2**, t can be made very long by choosing large k_m and k_1, or in other words, by making ΔZ small. Thus, the conclusion obtained in Sect. **2** that we can make the second measurement of position after a long time is still correct (7). Thus, there is no difference whatever in a velocity measurement of the type proposed by DIRAC for the case of particles satisfying the Dirac equation and for the case discussed in Sect. **2**.

5. – Conclusion.

We conclude then that the measurement proposed by DIRAC does not permit a clear manifestation of the new features of the velocity implied by the Dirac equation.

It is true that in the Dirac case, a particle with $V_z = + C$ may have either positive or negative momentum (as KOBA has pointed out), but the result of the experiment under discussion does not depend on the sign of the momentum and of the energy. Thus, KOBA's suggestion that in this experiment there should be a difference between the behavior of Dirac and non-Dirac particles resulting from the negative energies of the Dirac particles is seen to be inapplicable.

It must be pointed out here, however, that KOBA has shown correctly for a Dirac particle, how $\boldsymbol{x}$ and $\boldsymbol{v}$ can be defined simultaneously. But as we have seen in this paper, there is no need for a precise determination of z during the second measurement of position, in order to leave the particle in an eigenstate of the velocity. And it is evident that in the first measurement of position, the question of whether z and α_z commute is not relevant here. Thus, once again, we come to the conclusion that the special new possibilities implied by the Dirac equation are not demonstrated in this measurement.

We wish now to consider possible measurements that would show up the essential features of the velocity in the Dirac theory.

(1) To demonstrate the independence of $\boldsymbol{v}$ and $\boldsymbol{p}$ for the Dirac case (as we have seen this is the most new characteristic of the velocity in the Dirac theory), we might consider a statistical series of measurements for cases in

(7) In this regard, we disagree with KOBA who states in *Nuovo Cimento*, **3**, 214 (1956), that the second measurement of position must be made within a Compton time, etc.

which $V_z = C$ and $\overline{p}_z = 0$, while p_z fluctuates between large positive and negative values. This case is in fact just the one treated in Sect. **4**. We could measure the momentum by measuring the effective pressure due to a beam of particles in such a state. The independence of v and p would be clearly demonstrated if such a beam, moving with velocity, c, produced a zero pressure. Of course, this is only a hypothetical experiment, since we cannot actually obtain particles of negative energy; but it is no more hypothetical than the original experiment proposed by DIRAC.

(2) In the non-Dirac case, v can take on a continuous range of eigenvalues. Thus, a crucial test of the Dirac theory would be to measure v when p is well defined. But this will happen only if ψ takes the form of a plane wave. For this case, a direct measurement of the velocity with the aid of two successive position measurements is evidently not possible. There are, however, indirect consequences of the independence of velocity and momentum in the Dirac theory, which do not prove the Dirac expression for the velocity completely, but which provide evidence in its favor. One of these indirect consequences is just the magnetic moment of the electron, connected with its « spin ». Thus, from the well known Gordon decomposition (8) of the mean current density, we obtain

$$\boldsymbol{j} = ce(\psi^*\alpha\psi) \cong \frac{h}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{e\hbar}{2mc}\nabla\times(\psi^*\boldsymbol{\sigma}\psi)\,.$$

The first term in the above equation is also present in the non-Dirac case. The second term is an additional contribution to the current, which has the same effect as a magnetic moment. Thus, the observation of the value of the magnetic moment helps to verify the independence of $\boldsymbol{p}$ and $\boldsymbol{v}$ for Dirac particles.

(3) The commutation between $\boldsymbol{x}$ and $\boldsymbol{v}$ in the Dirac theory is very difficult to demonstrate experimentally in a clear way. For when x is well defined, then as we have seen, this commutation makes no essential difference in the motion of the particle. On the other hand, when x is not well defined, then there is no direct way of measuring the particle velocity. Perhaps some experiment utilizing wave packets of intermediate size could be designed to demonstrate the commutation of $\boldsymbol{x}$ and $\boldsymbol{v}$, but it would evidently be difficult to obtain precisely predictable results for such a case.

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(8) W. PAULI: *Handb. der Phys.*, **24**, 238 (1933).