# A Quantum Characterization of Classical Radiation*

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Any electromagnetic signal is representable, in the quantum mechanical description, by a suitable combination of photon states. We consider the question of which of the infinite number of possible combinations should correspond to a classical signal. The characteristic classical criterion we adopt is the indistinguishability of the radiation in two separate channels, whether it has been produced by independent sources or by a single source whose output is divided between the channels. For a quantum source a distinction is in general possible. We prove that the unique quantum state for which a distinction is not possible is the pure state characterized by Glauber as maximally coherent. The connection of this indistinguishability property with characteristic differences between classical and quantum measurements is emphasized.

## I. INTRODUCTION—THE PROBLEM

Sources of electromagnetic radiation emit photons. Nonetheless, a description of the emitted radiation by means of Maxwell's equations ignoring the photon aspect has proved adequate for most cases encountered in practice. Certainly the adequacy of a classical wave description of the electromagnetic radiation produced by macroscopic oscillations at radio frequencies is well established. Although the underlying photon picture is commonly accepted in this case there is no need to use it to account for any experimental result. On the other hand since the photon was introduced in connection with experiments on the emission and absorption of light from *microscopic* oscillators one expects that consideration of photons will be essential in the optical frequency range. For example, in experiments in which photons are counted the photon is an indispensable concept. However, the only role which the photon plays in most analyses is to carry discrete amounts of energy and momentum. As Franken has stated "it is possible to reproduce all the well-known results for the interaction of light and matter by postulating that quantum mechanics applies only to the matter and not to the light."[1] This is an affirmation of the utility of the conventional semiclassical

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[1] Attributed to P. Franken by Bradley and Foley (*1*).

radiation theory, in which the radiation is treated classically prior to its detection, at which point the appropriate number of photons is associated with the field. In this theory the electromagnetic field is taken to be a prescribed external classical field for the purpose of calculating transition matrix elements between atomic states.

The proper description of the state of the radiation field, as distinct from its interactions, entails considerably more quantum mechanics than the mere association of the photon with a packet of energy and momentum. In most optical experiments the proper quantum mechanical description is indistinguishable in practice from a classical wave description (*2*) in which the amplitude has a simple probability distribution. This apparent equivalence of two vastly different types of description results from the paucity of the set of measurements so far considered for the radiation field and not from an intrinsic unity of the descriptions. By making the field amplitude a stochastic variable with a simple probability distribution one can easily extend the classical radiation theory to provide an adequate description of the state of the radiation field for most of the optical experiments conducted thus far.[2]

To any classical description of the radiation field in terms of low order moments (correlation functions, etc.) there corresponds an infinite number of possible quantum mechanical states having the same moments. The insensitivity of this kind of classical description to the detailed quantum structure of the radiation state is largely responsible for its success. In view of the diversity in the possible choices of quantum states to correspond to a given classical state it is natural to ask whether one can find a simple physical criterion which makes the association between the two types of state unique. To make the problem simple we shall attempt to find such a criterion when the classical field is a prescribed nonstochastic function of time.

This question has been considered by Glauber (*3*, *4*). His criterion is to choose the quantum state of the electromagnetic field to be that produced by a classically prescribed current. In this prescription the current is assumed to suffer no reaction from the emitted radiation and is therefore taken to be a *c*-number in the quantum calculation of the radiation field. If one restricts the electromagnetic field to one mode, one can easily show that the quantum mechanical state vector for the field is given by the Glauber state $|\alpha\rangle$:

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}|n\rangle\alpha^n/\sqrt{n!}. \tag{1}$$

In Eq. (1) the vectors $|n\rangle$ denote states with $n$ photons in the selected mode, and $\alpha$ is a complex number related to the corresponding classical field amplitude.

[2] For a comprehensive survey of the relevant experiment and theory see (*2*).

This criterion is rooted in the method of generation of the field. One can ask for an alternative criterion which makes no reference to the generation of the field, but which refers instead only to operations on the field itself. Glauber has given a formal criterion of this kind; it also specifies that the quantum mechanical state vector is the $|\alpha\rangle$ of Eq. (1). Glauber has also given several other criteria intended to single out special quantum states to correspond with classical radiation fields; these criteria are satisfied by the states $|\alpha\rangle$ but there are additional states which satisfy them. His formal criterion is that the quantum state $|\alpha\rangle$ corresponding to a classical field in one mode with amplitude $\alpha$ is the eigenstate of the photon destruction operator $a$ for that mode:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2}$$

A criterion more closely related to possible measurements of the field is that there exist a function $\epsilon(x)$ such that each member $G^{(m)}(x_1, \cdots, x_{2m})$ of the infinite sequence of correlation functions factors as follows:

$$G^{(m)}(x_1, \cdots, x_{2m}) = \prod_{j=1}^{m} \epsilon^*(x_j)\epsilon(x_{m+j}). \tag{3}$$

This property of the $G$'s is used to define maximal coherence for a quantum mechanical state of the field. With this definition the Glauber states $|\alpha\rangle$ are maximally coherent, but they are not the only maximally coherent states.

The principal result of this paper is a simple physical criterion for the *unique* quantum state corresponding to a prescribed classical electromagnetic field. This criterion, which refers only to simple operations on the field itself, singles out the Glauber state, and thus provides an alternative characterization of it.

In Section II we shall present our simple physical criterion, the reasoning which suggested it, and some of its consequences. We shall show that the Glauber state is the only *pure* quantum state which satisfies our criterion. We shall then generalize our argument to apply to mixed quantum states possessing a $P$-representation (3). Finally in Section III we shall consider some simple examples which illustrate characteristic features of the theory of quantum measurement, features which naturally arise when we take into account the difference between measurement of classical and quantum systems—the criterion we propose is based on a consideration of the characteristic differences between measurements made on pure quantum and pure classical states.

## II. A SPLIT SOURCE VS. INDEPENDENT SOURCES

Suppose we consider a pair of radio transmitters $B$ and $C$ separated by many wavelengths of their basic carrier radio wave. We shall examine two distinct ways in which these transmitters may be driven. In case 1 the two transmitters are antennas fed by a common signal source $A$ connected to $B$ and $C$ by radio fre-

quency transmission circuits such that the radio signals from $B$ and $C$ will be identical, except possibly for an overall amplitude ratio which depends on the way the signal from $A$ is divided between the circuits carrying it to $B$ and $C$. Suppose the signal from $A$ is the carrier modulated at audio frequencies by a recording of Sibelius' Violin Concerto.

In case 2 the transmitters $B$ and $C$ are independent of each other; each has its own radio frequency oscillator and its own program of audio frequency modulation. Suppose that $B$ and $C$ both decide to play recordings of Sibelius' Violin Concerto; suppose the recordings are identical. Then according to classical physics there is no reason, in principle, why it should not be possible for the signals emanating from the transmitters $B$ and $C$ in case 2 to be identical to the signals emanating from the transmitters $B$ and $C$ in case 1. In classical physics the stochastic variations of the radio frequency oscillators and other components contributing to the signals can be imagined to be as small as we desire. Therefore, in classical physics we can suppose that for any prescribed nonstochastic signal produced by $A$ in case 1 it would be possible, by having suitable recordings available, to produce output signals from $B$ and $C$ in case 2 that would be *indistinguishable* from the output signals from $B$ and $C$ in case 1. (We exclude wholly stochastic signals because for these it is easy to distinguish between cases 1 and 2. Indeed, consider stochastic sources with zero average field. If a single such source is split as in case 1, the average product of the fields in channels $B$ and $C$ is always positive, but if the fields in these channels are produced by statistically independent sources, the average of the product is zero.)

If the cases 1 and 2 are set up classically as described above, then it is impossible to determine which case actually obtains if we are not permitted to examine the sources $A$, $B$, and $C$ and their connecting links; that is, if we are confined to measurements on the radiation emitted by the system. On the other hand considering cases 1 and 2 with quantum sources we might suppose that it would *always* be possible to tell the difference between the two cases if all possible measurements on the output radiation are allowed to us. This supposition is an error, and it is the error that this paper is about. Note that we are explicitly including *all* possible measurements on the output radiation; correlation measurements relating numbers of photons produced by $B$ to the number produced by $C$ are *included*. In the classical situation we have described so far such correlation measurements give no new physical information: if the signals from $B$ and $C$ are nonstochastic, *local* measurements, i.e., those made on $B$ and on $C$ separately, will suffice to predict the results of all possible correlations between the output of $B$ and the output of $C$. However, in quantum mechanics correlation measurements may be needed to complete the specification of pure states (5). In Section III we give an explicit example of two pure quantum states which are indistinguishable if we are allowed to perform only *local* measurements on

them, but which are distinguishable when correlation experiments are added to our arsenal of possible measurements.

The erroneous argument that led us to suppose that case 1 can always be distinguished from case 2 if source $A$ is a quantum source is the following. We imagine that source $A$ produces a pure quantum state containing photons of a definite carrier frequency. We shall neglect the modulation of the carrier; that can be inserted by the reader by making the coefficients occurring in our expressions time dependent. Then the most general pure state possible at source $A$ has the following form:

$$|\psi_A\rangle = \sum_{m=0}^{\infty} C_{Am} |m\rangle_A , \tag{4}$$

where the vector $|m\rangle_A$ represents a state in which $m$ carrier frequency photons are present at source $A$, and the $C_{Am}$ are arbitrary complex numbers that satisfy the normalization condition

$$\sum_{m=0}^{\infty} C_{Am} C_{Am}^* = 1. \tag{5}$$

We suppose that the contents of source $A$, once they are generated by $A$, are divided between $B$ and $C$ in a definite way. We suppose that there is an amplitude $\mu$ for a photon in $A$ to be sent to $B$, and an amplitude $\nu$ for that photon in $A$ to be sent to $C$. For example, the amplitudes could both be $\sqrt{1/2}$ if the splitting of the signal from $A$ were accomplished by an ideal half-silvered mirror. We then concluded that the splitting action would obviously produce correlations between the contents of $B$ and the contents of $C$: if a photon goes to $B$, it cannot go to $C$, and vice versa. Such correlated states are described more fully in Section III.

But in case 2 we have independent quantum sources at $B$ and $C$, so we expect that the quantum state produced by such sources must factorize; there should be no correlations between the radiation produced at $B$ and the radiation produced at $C$. Clearly that expectation cannot be wrong; it is what we mean by independent sources. Hence we made the erroneous conclusion that case 1, which we supposed to produce a correlated quantum state, could always be distinguished in principle from case 2, which must produce uncorrelated states. If that were so then no matter what quantum source we had at $A$; that is, whatever the coefficients $C_{Am}$, then such a source driving $B$ and $C$ would always be distinguishable by its output at $B$ and $C$ from any independent sources driving $B$ and $C$: the classical indistinguishability would never obtain.

We then tested this prediction on a Glauber state source at $A$ and found it to be false: there are no correlations in the $B$-$C$ quantum state if the state $|\psi_A\rangle$ s a Glauber state. It occurred to us that possibly the Glauber state is the only ipure quantum state possessing this indistinguishability property: we proved that

it is. Then we wondered whether there exist other quantum states having the indistinguishability property if we enlarge the class of quantum states to include statistical mixtures specified by a density matrix. Again the Glauber state is the only quantum state which has this classical characteristic.

We now present three[3] things: (1) a demonstration that the Glauber state at $A$ produces an uncorrelated state (a state which factorizes) at $B$-$C$; (2) a proof that the only *pure* quantum state at $A$ that can produce a factorizing state at $B$-$C$ is the Glauber state; and (3) an argument to show that the only quantum state that produces a factorized state at $B$-$C$ is the Glauber state.

Up to this point we have referred to a signal source $A$ and transmitter $B$ and $C$. For generality we shall drop this hardware description and refer henceforth to channels $A$, $B$ and $C$. Each channel will be taken to correspond to one mode of the electromagnetic field. The modes $B$ and $C$ are assumed orthogonal. This corresponds to the separation of the transmitters introduced previously.[4]

Since the $n$-photon state $|\, n\rangle_A$ can be written

$$|\, n\rangle_A = [(a_A^\dagger)^n/(n!)^{1/2}]|\, 0\rangle_A \tag{6}$$

where $|\, 0\rangle_A$ is the no-photon state and $a_A^\dagger$ is the usual photon creation operator for channel $A$, an equivalent expansion to (4) is

$$|\psi_A\rangle = \sum_{n=0}^{\infty} d_{nA}(a_A^\dagger)^n|\, 0\rangle_A\,. \tag{7}$$

The most general states generated by independent sources in channels $B$ and $C$ can be written similarly:

$$|\, \psi_B\rangle = \sum_{n=0}^{\infty} d_{nB}(a_B^\dagger)^n\,|\, 0\rangle_B\,;\; |\, \psi_C\rangle = \sum_{n=0}^{\infty} d_{nC}(a_C^\dagger)^n\,|\, 0\rangle_C\,. \tag{8}$$

Since the coefficients $C_{mA}$, $C_{mB}$, $C_{mC}$ are normalized according to Eq. (5), it follows that the coefficients $d_{mA}$, $d_{mB}$, $d_{mC}$ are such that the functions

$$f_j(z) = \sum_{n=0}^{\infty} d_{nj}z^n \tag{9}$$

are entire functions. The most general state vector $|\, \psi\rangle_2$ for case 2 is

$$|\, \psi\rangle_2 = f_B(a_B^\dagger)f_C(a_C^\dagger)\,|\, 0\rangle_{BC} \tag{10}$$

[3] These three results were presented in a paper at the Conference on the Physics of Quantum Electronics, Puerto Rico, June 1965; a brief account of items (1) and (2) has appeared in the proceedings of that conference (*6*).

[4] Other examples of channels are separate light beams or transmission lines. Spatial separation is not essential as long as the radiation modes are orthogonal, e.g. orthogonal polarizations in the same beam. In this context a "local" measurement is one made on one channel.

where the no-photon state for channels $B$ and $C$ may be written as

$$|0\rangle_{BC} = |0\rangle_B \otimes |0\rangle_C , \tag{11}$$

the direct product of the no-photon states for channels $B$ and $C$.

For case 1, the source $A$ produces a state

$$|\psi_A\rangle = f_A(a_A^\dagger)\,|0\rangle_A \tag{12}$$

each photon of which is projected by the splitter with amplitude $\mu$ or $\nu$ into the modes of channels $B$ and $C$ respectively. This splitting is expressed by defining the operator

$$a_A^\dagger = \mu a_B^\dagger + \nu a_C^\dagger \tag{13}$$

which is readily verified to act as a creation operator for the desired one-photon superposition state. The associated annihilation operator is

$$a_A = \mu^* a_B + \nu^* a_C . \tag{14}$$

From the Bose commutation relations for the operators $a_B^\dagger$, $a_B$ and $a_C^\dagger$, $a_C$ and the assumed orthogonality of modes $B$ and $C$ one finds

$$[a_A , a_A^\dagger] = 1 \tag{15}$$

provided

$$|\mu|^2 + |\nu|^2 = 1. \tag{16}$$

Since $A$ sends its photons to $B$ and $C$, we may consistently relate the vacuum states by the direct product:

$$|0\rangle_A = |0\rangle_B \otimes |0\rangle_C . \tag{17}$$

We may now express the condition for the indistinguishability of cases 1 and 2 as the equality of the state vectors of the radiation field for the two cases:

$$|\psi_A\rangle \equiv |\psi\rangle_1 = |\psi\rangle_2 . \tag{18}$$

This is clearly a sufficient condition. In Section III we shall show that (18) (except for a trivial phase factor) is also a necessary condition for the indistinguishability of cases 1 and 2 if *all* measurements are allowed. Equation (18) requires that

$$f_A(\mu a_B^\dagger + \nu a_C^\dagger) = f_B(a_B^\dagger) f_C(a_C^\dagger). \tag{19}$$

If this equation, $f_A$ , $f_B$ , and $f_C$ could, *a priori*, be different functions characterizing the different sources. The solutions of this (operator) functional equation determine explicitly the admissible pure quantum states (in the form specified by Eq. (12)) for which cases 1 and 2 are equivalent.

First let us verify that the Glauber coherent states satisfy Eq. (19). The function $f_\alpha$ for a Glauber state $|\alpha\rangle$ with amplitude $\alpha$ may be written

$$f_\alpha(a^\dagger) = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^\dagger). \tag{20}$$

Then the split field from $A$ is gotten from this with $a^\dagger = a_A^\dagger$, see Eq. (13):

$$f_\alpha(\mu a_B^\dagger + \nu a_C^\dagger) = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\mu\alpha a_B^\dagger + \nu\alpha a_C^\dagger). \tag{21}$$

Using Eq. (16), this is equal to

$$[\exp(-\tfrac{1}{2}|\alpha\mu|^2)\exp(\alpha\mu a_B^\dagger)][\exp(-\tfrac{1}{2}|\alpha\nu|^2)\exp(\alpha\nu a_C^\dagger)] = f_{\mu\alpha}(a_B^\dagger)f_{\mu\alpha}(a_C^\dagger). \tag{22}$$

The righthand side corresponds to Glauber states in the channels $B$ and $C$ with amplitudes $\mu\alpha$ and $\nu\alpha$ respectively. In terms of the state vectors, (22) is equivalent to[5]:

$$|\alpha\rangle_A = |\mu\alpha\rangle_B \otimes |\nu\alpha\rangle_C . \tag{22a}$$

Thus a sufficient condition for the equivalence of cases 1 and 2 is that the three sources each produce a Glauber coherent state with respective amplitudes $\alpha$, $\mu\alpha$, and $\nu\alpha$. We next show that this condition is also necessary; these states are the unique quantum states with this property. It will then follow that for every other quantum state, no equivalence between cases 1 and 2 is possible: some measurable distinction can in principle always be made.

We proceed to obtain the explicit solution of the functional equation (19). From Eq. (19) one has

$$f_A(0) = f_B(0)f_C(0). \tag{23}$$

Define $F(z) = f(z)/f(0)$ and divide Eq. (19) by Eq. (23) to get

$$F_A(\mu a_B^\dagger + \nu a_C^\dagger) = F_B(a_B^\dagger)F_C(a_C^\dagger). \tag{24}$$

This functional equation is equivalent to Eq. (19) and has the added convenience of the boundary conditions $F_i(0) = 1$, $i = A, B, C$. Setting in turn $a_C^\dagger = 0$, $a_B^\dagger = 0$ in Eq. (24) one obtains

$$F_A(\mu a_B^\dagger) = F_B(a_B^\dagger), \tag{25}$$

$$F_A(\nu a_C^\dagger) = F_C(a_C^\dagger). \tag{26}$$

[5] We remark that although the state $|\alpha\rangle_A$ defined by Eq. (22a) is an eigenstate of the operator $a_A = \mu^* a_B + \nu^* a_C$, this is not a unique way of characterizing it because (unlike the non-degenerate eigenstates of $a_B$ or $a_C$) there are an infinite number of eigenstates of $a_A$. Indeed, any state of the form $|C_1\alpha\rangle_B \otimes |C_2\alpha\rangle_C$ with $C_1$, $C_2$ arbitrary numbers is an eigenstate of $a_A$. But only with $C_1 = \mu$, $C_2 = \nu$ does one get the unique eigenstate corresponding to the splitting with relative amplitudes $\mu$ and $\nu$. It has the eigenvalue $\alpha$.

On substituting these into Eq. (24) one gets a functional equation involving only one unknown function, $F_A$ :

$$F_A(\mu a_B^\dagger + \nu a_C^\dagger) = F_A(\mu a_B^\dagger)F_A(\nu a_C^\dagger). \tag{27}$$

This is a familiar functional equation in the theory of one-parameter semigroups (*7*). With the above boundary condition, it has the unique solution

$$F_A(a^\dagger) = \exp(\alpha a^\dagger), \tag{28}$$

where $\alpha$ is an arbitrary complex parameter. From Eq. (25) and Eq. (26), we have that

$$F_B(a_B^\dagger) = \exp(\alpha\mu a_B^\dagger), \tag{29}$$

$$F_C(a_C^\dagger) = \exp(\alpha\nu a_C^\dagger). \tag{30}$$

Thus the operator functions $f_i(a^\dagger)$ for each of the sources are, to within a constant factor $f_i(0)$, precisely those which produce Glauber coherent states (see Eq. (20)). The constant factors do not affect the state and may be determined by normalizing the states. (They are readily found to be $f_A(0) = \exp(-\frac{1}{2}|\alpha|^2)$, $f_B(0) = \exp(-\frac{1}{2}|\alpha\mu|^2)$, and $f_C(0) = \exp(-\frac{1}{2}|\alpha\nu|^2)$). This completes the proof of uniqueness.

We now relax the requirement of *pure* quantum states and ask if there are more general quantum mechanical states, e.g., statistical mixtures of pure quantum states for which cases 1 and 2 are equivalent. Mixed or impure states cannot be expanded in the form specified by Eq. (4) with fixed expansion coefficients $C_n$ . Rather they are the quantum version of what one gets classically when the coefficients $C_n$ in the expansion are random variables, i.e., noise. In quantum mechanics such states are represented by a density operator $\rho$. The analysis here will make use of an expansion due to Glauber (*3*, *4*) and Sudarshan (*8*) which represents the density operator for an arbitrary state of the radiation field as a sum over suitably weighted coherent state projection operators:

$$\rho = \int d^2\alpha P(\alpha) \,|\, \alpha\rangle\langle\alpha \,|. \tag{31}$$

Here $P(\alpha)$ is a real (but not necessarily positive) valued (generalized) weight function of the complex argument $\alpha$; the integration is over the 2-dimensional complex $\alpha$-plane. Thus we write for the radiation fields due to the sources $B$ and $C$

$$\rho_B = \int d^2\alpha_B P_B(\alpha_B) \,|\, \alpha_B\rangle\langle\alpha_B \,|, \tag{32}$$

$$\rho_C = \int d^2\alpha_C P_C(\alpha_C) \,|\, \alpha_C\rangle\langle\alpha_C \,|. \tag{33}$$

In this notation $\alpha_B$ , say, labels both the coherent state vector for channel $B$, $|\,\rangle_B$ , and the eigenvalue of the annihilation operator $a_B$ acting on this state.

The radiation field produced by the source $A$ when split may also be expanded in this form,

$$\rho_A = \int d^2\alpha_A P(\alpha_A) \mid \alpha_A\rangle\langle\alpha_A \mid. \tag{34}$$

with

$$\mid \alpha_A\rangle = \exp(-\tfrac{1}{2}\mid \alpha \mid^2) \sum_{n=0}^{\infty} \alpha^n (\mu a_B^\dagger + \nu a_C^\dagger)^n \mid 0\rangle_{BC}/n!. \tag{35}$$

Since

$$a_A \mid \alpha_A\rangle = \alpha \mid \alpha_A\rangle \tag{35a}$$

we drop the subscript $A$ on $\alpha_A$, the $\alpha$ itself indicating quantities associated with the split field $A$. Similarly in Eq. (22a),

$$\mid \alpha\rangle_A = \mid \mu\alpha\rangle_B \otimes \mid \nu\alpha\rangle_C \equiv \mid \mu\alpha, B; \nu\alpha, C\rangle; \tag{35b}$$

the indices $B$ and $C$ are dispensable since the channels are identifiable by the parameters $\mu$ and $\nu$. With this notation Eq. (34) can be written

$$\rho_A = \int d^2\alpha P_A(\alpha) \mid \mu\alpha; \nu\alpha\rangle\langle\nu\alpha; \mu\alpha \mid. \tag{36}$$

The functional equation analogous to Eq. (19) which expresses the equivalence of cases 1 and 2 is now in terms of the density operators

$$\rho_A = \rho_B \rho_C . \tag{37}$$

In order to solve this equation for the $\rho$'s or their associated weight functions, the $P$'s, we arrange to have the domain of integration to be the same on both sides. This can be done by defining a new function

$$P_A(\alpha_B, \alpha_C) \equiv \int d^2\alpha P_A(\alpha)\delta^{(2)}(\alpha_B - \mu\alpha)\delta^{(2)}(\alpha_C - \nu\alpha), \tag{38}$$

where $\delta^{(2)}$ is a 2-dimensional Dirac $\delta$-function of its complex argument. Then

$$\begin{aligned}
&\iint d^2\alpha_B\, d^2\alpha_C P_A(\alpha_B, \alpha_C) \mid \alpha_B ; \alpha_C\rangle\langle\alpha_C ; \alpha_B \mid \\
&= \int d^2\alpha P_A(\alpha) \left[ \iint \delta^{(2)}(\alpha_B - \mu\alpha)\delta^{(2)}(\alpha_C - \nu\alpha) \mid \alpha_B ; \alpha_C\rangle\langle\alpha_C ; \alpha_B \mid d^2\alpha_B\, d^2\alpha_C \right] \\
&= \int d^2\alpha P_A(\alpha) \mid \mu\alpha; \nu\alpha\rangle\langle\nu\alpha; \mu\alpha \mid \\
&= \rho_A.
\end{aligned} \tag{39}$$

On the other hand, by direct substitution from Eq. (32) and Eq. (33)

$$\rho_B\rho_C = \iint d^2\alpha_B\, d^2\alpha_C P_B(\alpha_B)P_C(\alpha_C)\ |\ \alpha_B\,;\,\alpha_C\rangle\langle\alpha_C\,;\,\alpha_B\ |. \tag{40}$$

Imposing the condition of Eq. (37), we may equate Eq. (40) to the first representation of $\rho_A$ in Eq. (39). And since the basis states are the same in both integrals it is clear that condition Eq. (37) will be satisfied if

$$P_A(\alpha_B\,,\,\alpha_C) = P_B(\alpha_B)P_C(\alpha_C). \tag{41}$$

The definition Eq. (38) for $P_A(\alpha_B\,,\,\alpha_C)$ yields a more explicit form for it when one carries out one of the indicated integrations:

$$P_A(\alpha_B\,,\,\alpha_C) = P_A(\alpha_B/\mu)\delta^{(2)}(\alpha_C - \alpha_B[\nu/\mu])/|\,\mu\,|^2. \tag{42}$$

Thus the functional equation for the unknown weight functions $P_i$ takes the form

$$P_B(\alpha_B)P_C(\alpha_C) = P_A(\alpha_B/\mu)\delta^{(2)}(\alpha_C - \alpha_B[\nu/\mu])/|\,\mu\,|^2. \tag{43}$$

The solution of this equation is

$$P_A(\alpha_A) = \delta^{(2)}(\alpha_A - \alpha),\ \alpha \text{ arbitrary}, \tag{44}$$

$$P_B(\alpha_B) = \delta^{(2)}(\alpha_B - \mu\alpha_A), \tag{45}$$

$$P_C(\alpha_C) = \delta^{(2)}(\alpha_C - \nu\alpha_C). \tag{46}$$

This can be seen as follows: In Eq. (43) fix $\alpha_B$ at some value $\alpha_B'$ for which $P_A(\alpha_B'/\mu) \neq 0$. Then considering both sides as functions of $\alpha_C$ we see that $P_C$ must be proportional to $\delta^{(2)}(\alpha_C - \alpha_B'[\nu/\mu])$. If there were another value $\alpha_B''$ for which $P_A(\alpha_B''/\mu) \neq 0$, we would obtain contradictory solutions for $P_C$. Hence there is at most one value of $\alpha_B$ for which $P_A(\alpha_B/\mu) \neq 0$. From the symmetry of Eq. (41) in $\alpha_B$ and $\alpha_C$ it is clear that similarly $P_B$ must be proportional to $\delta^{(2)}(\alpha_B - \alpha_C'[\mu/\nu])$ and that $\alpha_C'$ is unique. Since both $P_B$ and $P_C$ are $\delta^{(2)}$ functions, it follows from Eq. (43) that $P_A$ is also a $\delta^{(2)}$ function.

With the above solution for the $P$'s the corresponding $\rho$'s each reduce to a projection operator for a pure coherent state with amplitudes $\alpha$, $\mu\alpha$, and $\nu\alpha$ respectively, thus extending the theorem previously proven for pure states to statistical mixtures of quantum states. Although this result seems highly plausible for quantum noise sources by analogy with what one expects from classical random sources, the proof given here is not rigorous in two respects:

1. There is some question[6] as to the validity of the $P$ representation for all possible quantum states of radiation.

2. The coherent states are not linearly independent; they form an over-

[6] See the references cited in (*2*), (*4*), (*9*), and (*10*).

complete set (*3*). Hence the standard argument for equating coefficients in sums over states needs further justification.[7]

## III. MEASUREMENTS ON TWO-CHANNEL SYSTEMS

In this section we consider various measurements that can be made on a two-channel system and the information we obtain thereby. We shall see that the set of all measurements determines the state vector up to a phase if the system is in a pure state (or more generally it determines the density matrix of the system), while the set of local measurements rarely determines these.

Suppose that the basis states for channel $B$ are specified by vectors $|\, r\rangle_B$ and that the basis states for channel $C$ are specified by vectors $|\, s\rangle_C$. Then the most general pure state for the two-channel system is

$$|\, \psi\rangle = \sum_{rs} C_{rs}\, |\, r\rangle_B \otimes |\, s\rangle_C . \tag{47}$$

According to the general principles of quantum mechanics (*12*) we may determine, by repeated measurements on identically prepared systems, the expectation value of any Hermitian operator. Consider the non-Hermitian transition operators $T_{tu,rs}$:

$$T_{tu,rs} \equiv \{|\, t\rangle_B \otimes |\, u\rangle_C\}\{\langle s\,|_C \otimes \langle r\,|_B\}. \tag{48}$$

We can write $T_{tu,rs}$ as a linear combination of two Hermitian operators:

$$T_{tu,rs} = H_{tu,rs} + iA_{tu,rs}\,, \tag{49}$$

where

$$\begin{aligned} H_{tu,rs} &= \tfrac{1}{2}[|\, tu\rangle_{BC}\langle sr\,| + \tfrac{1}{2}[|\, rs\rangle_{BC}\langle ut\,|], \\ A_{tu,rs} &= -\tfrac{1}{2}i[|\, tu\rangle_{BC}\langle sr\,| + \tfrac{1}{2}i[|\, rs\rangle_{BC}\langle ut\,|]. \end{aligned} \tag{50}$$

Since we may determine the expectation values of each of the $H$'s and $A$'s, we may determine the expectation values of the $T$'s. But those expectation values form the density matrix for the system:

$$\langle T_{tu,rs}\rangle = {}_{CB}\langle tu\,|\,\rho\,|\,rs\rangle_{BC} = \rho_{tu,rs}\,. \tag{51}$$

The density operator $\rho$ is defined by

$$\rho = \sum_i W_i\, |\, \psi_i\rangle\langle\psi_i\,|, \tag{52}$$

where $W_i$ is the probability of finding the system in the pure state $|\, \psi_i\rangle$. Hence the set of measurements determining the $H$'s and $A$'s determines the density matrix $\rho$ (*13*).

If the system is in a pure state $|\, \psi\rangle$, $\rho$ reduces to the projection operator for

[7] Proofs of the uniqueness of Glauber states, free of these objections, have recently been communicated to us, first by Glauber and Titulaer (*10*) and then by T. F. Jordan (*11*).

that state:

$$\rho = |\psi\rangle\langle\psi| \tag{53}$$

and

$$\rho_{tu,rs} = \langle\psi | tu\rangle_{BC\ CB}\langle sr | \psi\rangle = C_{rs}C^*_{tu}\,. \tag{54}$$

The eigenvalue equation $\rho|\psi\rangle = |\psi\rangle$ which implies

$$\sum_{rs} \rho_{tu,rs}C_{rs} = C_{tu} \tag{55}$$

determines the coefficients $C_{rs}$ up to a common factor which, in virtue of normalization, is of the form $e^{i\phi}$. This shows that if every possible measurement for case 1 yields the same result as for case 2, then the state vectors for these two cases must be the same, as stated in connection with Eq. (18).

Suppose now that we are permitted local measurements only, that is, measurements on either channel $B$ or channel $C$. An operator representing a measurement on channel $B$ alone must be a direct product of an operator on channel $B$ and the unit operator on channel $C$. All such operators can be obtained as linear combinations of the following basic set $T^B_{tr}$, $T^C_{us}$:

$$\begin{aligned} T^B_{tr} &= \sum_s T_{ts,rs} = |t\rangle_{B\ B}\langle r| \otimes \sum_s |s\rangle_{C\ C}\langle s|\,; \\ T^C_{us} &= \sum_r T_{ru,rs}\,. \end{aligned} \tag{56}$$

The operators $T^B_{tr}$ and $T^C_{us}$ can be written as linear combinations of local Hermitian operators:

$$T^B_{tr} = H^B_{tr} + iA^B_{tr}\,, \qquad H^B_{tr} = \sum_s H_{ts,rs}\,, \qquad A^B_{tr} = \sum_s A_{ts,rs}\,, \tag{57}$$

and similarly for $C$. Therefore the set of all local measurements will enable us to determine the values of the single channel density matrices $\rho^B_{tr}$, $\rho^C_{us}$:

$$\langle T^B_{tr}\rangle \equiv \rho^B_{tr}\,, \qquad \langle T^C_{us}\rangle \equiv \rho^C_{us}\,. \tag{58}$$

If we know that the density matrix for the whole system factors,

$$\rho_{tu,rs} = F_{tr}G_{us}\,, \tag{59}$$

for all $t$, $r$, $u$, $s$, it is easily shown that

$$\rho_{tu,rs} = \rho^B_{tr}\rho^C_{us}\,. \tag{60}$$

Therefore, in this case, local measurements *do* determine the complete density matrix for the system. In general, however, the density matrix for the whole system does not factor; nor in general is it possible to determine it from the values of $\rho^B_{tr}$ and $\rho^C_{us}$. Von Neumann[8] has proved that there are infinitely many

[8] See *13*, p. 428.

density matrices for the whole system consistent with given density matrices for the channels $B$ and $C$ unless one of these corresponds to a pure state. In this case the complete density matrix is unique and factors, Eq. (60).

As an illustration of von Neumann's theorem we now give an example in which two different complete density matrices yield the same single channel density matrices. Moreover, both of the complete density matrices represent pure states; one, Eq. (61), corresponds to case 1. The channel density matrices for our example both represent mixed states as required by von Neumann's theorem.

We restrict ourselves to states such that at most one photon can be in a given channel. Consider the following two states:

$$|\psi\rangle = \sqrt{\tfrac{1}{2}}\,|0_B0_C\rangle + \tfrac{1}{2}\,|1_B0_C\rangle + \tfrac{1}{2}\,|0_B1_C\rangle, \tag{61}$$

$$\begin{aligned}|\phi\rangle = {} & \sqrt{\tfrac{5}{8}}\,|0_B0_C\rangle + \sqrt{\tfrac{1}{80}}\,[3+i]\,|1_B0_C\rangle \\ & + \sqrt{\tfrac{1}{80}}\,[3+i]\,|0_B1_C\rangle + \sqrt{\tfrac{1}{40}}\,[2-i]\,|1_B1_C\rangle.\end{aligned} \tag{62}$$

Because both $|\psi\rangle$ and $|\phi\rangle$ are symmetric in each channel $\rho_\phi{}^B = \rho_\phi{}^C$ and $\rho_\psi{}^B = \rho_\psi{}^C$. Moreover these two particular states have been constructed so as to make all the single channel density matrices identical

$$\rho_\phi{}^B = \rho_\psi{}^B = \rho_\psi{}^C = \rho_\phi{}^C = \begin{pmatrix} \tfrac{3}{4} & \tfrac{1}{2}\sqrt{\tfrac{1}{2}} \\ \tfrac{1}{2}\sqrt{\tfrac{1}{2}} & \tfrac{1}{4} \end{pmatrix}. \tag{63}$$

Hence in this example all local measurements on the two states will yield identical results. But clearly a nonlocal measurement such as a coincidence count for channels $B$ and $C$ would yield different results.

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