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Nonlocal Effects in Classical and Quantum Theories*

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Quantum effects of electromagnetic potentials are analyzed further. It is shown that other theories, both classical and quantum mechanical, exhibit analogous effects. Examples from classical general relativity, Yang–Mills theory, and quantum field theory are discussed. It is concluded that effects of this kind are not necessarily quantum mechanical in nature and not necessarily related to the appearance of non-gauge-invariant potentials in the formulation of the theory.

INTRODUCTION

Some years ago it was pointed out *(1)* that electromagnetic potentials seemed to have peculiar effects in the quantum domain. It was shown that the motion of a charged particle taking place in a multiply-connected region free of electromagnetic fields but enclosing a flux depends on the enclosed flux. These effects had been confirmed experimentally *(2)*.

In subsequent discussions it was taken for granted that the effect is necessarily quantum mechanical and involves either local interaction with potentials or nonlocal interactions with fields.

In this article we analyze further the essential features of the effect. We investigate other theories in which analogous effects are present and demonstrate that such effects are not necessarily quantum mechanical in nature.

Finally we show that it is possible to formulate local theories in terms of gauge-invariant quantities in which the same effects are present.

I

In this section we discuss the potential effects in multiply-connected regions exhibited by a general gauge field.[1]

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[1] The idea that such effects may be present in general gauge fields had also been suggested by others, e.g., B. S. De Witt [*Phys. Rev.* **125**, 2189 (1962)] and G. Carmi (personal communication).

We consider a field $\varphi(x)$, which belongs to a representation of a continuous group of transformations G, and which satisfies linear equations of motion. We assume the usual type of interaction between $\varphi(x)$ and the gauge potentials $B_\mu{}^i$ associated with G, namely one in which covariant derivatives of $\varphi(x)$ replace ordinary derivatives in the free equations of motion for $\varphi(x)$. Examples of such interactions are charged fields interacting with the electromagnetic field, tensor fields in interaction with the gravitational field, nucleons interacting with the Yang-Mills field (*3*), etc.

Let us consider a setup analogous to that of the electromagnetic case (*1*). A flux of the gauge field $F^i_{\mu\nu}$ is enclosed in a region R, and a packet of the field $\varphi(x)$ moves in a multiply-connected region R' surrounding R. Let Φ_0 be the total class of solutions (not necessarily single-valued) in R' of the equations of motion for $\varphi(x)$, in the case in which the field $F^i_{\mu\nu}$ is zero everywhere.

When $F^i_{\mu\nu}$ is not zero in R (but zero in R') the solution is

$$\varphi(x) = D(a)\varphi_0(x), \tag{1}$$

where $\varphi_0(x)$ represents a suitable non-single-valued subclass of Φ_0, $D(a)$ stands for a gauge transformation with space–time-dependent parameters $a(x)$ and $D(a)\varphi_0(x)$ is single-valued over the whole R' ($a(x)$ is in general multiple-valued). To see this we compute the covariant derivative $\varphi_{.\mu}$. Let α be the coupling constant of the interaction of φ with $B^i{}_\mu$, $L^i{}_j$ a matrix derived from the multiplication table of the group elements $c = F(a, b)$ given by

$$L^i{}_j(b) = \frac{\partial F^i}{\partial a^j}(a, b)\Big|_{a=0},$$

and T_i the infinitesimal generators of the group, then

$$\begin{aligned} \varphi_{.\mu} &= D(a)\varphi_{0,\mu} + D(a)_{,\mu}\varphi_0 + \alpha B^i{}_\mu T_i D(a)\varphi_0 \\ &= D(a)\varphi_{0,\mu} + a^k_{,\mu} L_k^{-1i}(a) T_i D(a)\varphi_0 \\ &\qquad + \alpha B^i{}_\mu T_i D(a)\varphi_0 \text{ [Ref.(4)]}. \end{aligned}$$

Since

$$\alpha B^i{}_\mu(x) = -a^k_{,\mu} L_k^{-1i}(a) \tag{2}$$

in the regions where the field $F^i_{\mu\nu}$ vanishes, we have finally

$$\varphi_{.\mu} = D(a)\varphi_{0,\mu}.$$

This result assures that $\varphi(x)$ is the solution because φ_0 is chosen to be a solution of the free case.

Let us now consider in more detail the effect of the field $F^i_{\mu\nu}$ enclosed in R on the motion of a packet of field φ in R'. In addition to the change in φ that would occur if its motion were free, there is an extra gauge transformation on φ with parameters given by Eq. (2). When the orbit is closed the added change of φ will be a constant, finite gauge transformation, independent of the shape of the packet and of its orbit. The parameter of this gauge transformation is obtained by integrating Eqs. (3) and (2) and is given by (5)

$$D = T \exp\left(-\alpha \oint T_i B^i_{\mu} dx^{\mu}/ds\, ds\right) = e^{b^i T_i}, \tag{4}$$

where T is an ordering operator with respect to the parameters that labels the points along the orbit of φ. It follows from Eq. (4) that the resulting effect on φ due to the presence of $F^i_{\mu\nu}$ depends on the global properties of the potentials $B_{\mu}{}^i$.

In the following sections we discuss consequences of the above results in different gauge theories of physical interest, including classical as well as quantum theories.

II

Consider the classical gravitational field and let it interact with another field (i.e., tensor or spinor) in the way described in [II] [see Ref. (4), p. 700]. In this case the gauge field is associatedwith the Lorentz group.

To begin with we discuss the effect of the gravitational field on the parallel displacement of a vector in a multiply-connected region. For simplicity we restrict ourselves to a two-dimensional case where the manifold is a cone. The curvature vanishes everywhere except in the vertex. The vector is carried around a closed orbit encircling the vertex. After the orbit is completed the direction of the vector will be rotated through an amount equal to the flux of curvature enclosed by the orbit of the vector (6).

As in the general example discussed in I, the above change in the direction of the vector is due to a global effect of the enclosed curvature, that cannot be traced locally in an invariant (i.e., coordinate-independent) way.

Next we discuss the effect of the curvature of space time on the interference of electromagnetic waves. An electromagnetic wave is split in two packets; one of the packets is made to move around a close path and then recombined with the other packet. Suppose first that the path encloses no curvature, the energy distribution is then

$$\epsilon = \mathbf{F}\cdot\mathbf{F}^*, \tag{5}$$

where

$$\mathbf{F} = \mathbf{E} + i\mathbf{H}; \qquad \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2; \qquad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2,$$

so

$$\epsilon = \epsilon_1 + \epsilon_2 + 2(\mathbf{E}_1\cdot\mathbf{E}_2 + \mathbf{H}_1\cdot\mathbf{H}_2). \tag{6}$$

Here ϵ_1 and ϵ_2 are the energies of each separate wave and $2(\mathbf{E}_1\mathbf{E}_2 + \mathbf{H}_1\mathbf{H}_2)$ is the interference term.

Suppose now that the path encloses a bound region of curved space–time. In this case the distribution of energy is given by

$$\epsilon' = (\mathbf{F}_1 + \mathbf{F}_2')\cdot(\mathbf{F}_1^* + \mathbf{F}_2^{*\prime}). \tag{7}$$

where $\mathbf{F}_2' = \mathfrak{R}\mathbf{F}_2$ [see (7)], and $\mathfrak{R}$ is a three-dimensional complex rotation. The six real parameters which define the three complex angles of this rotation are the parameters of the Lorentz transformation undergone by the vector potential A_μ when it is transported along the same orbit.

If we write $\mathfrak{R} = \mathfrak{N} + i\mathfrak{M}$, with $\mathfrak{N}$ and $\mathfrak{M}$ real and use the orthogonality condition

$$\mathfrak{R}\tilde{\mathfrak{R}} = 1, \tag{8}$$

we deduce

$$\mathfrak{N}\tilde{\mathfrak{N}} - \mathfrak{M}\tilde{\mathfrak{M}} = 1, \tag{9}$$

$$\mathfrak{N}\tilde{\mathfrak{M}} + \mathfrak{M}\tilde{\mathfrak{N}} = 0. \tag{10}$$

Using (9) and (10) we can write the energy distribution (7) in the form

$$\begin{aligned}\epsilon' = \epsilon_1 + \epsilon_2 + 2(\mathfrak{M}\mathbf{E}_2)\cdot(\mathfrak{M}\mathbf{E}_2) + 2(\mathfrak{M}\mathbf{H}_2)\cdot(\mathfrak{M}\mathbf{H}_2) + 4(\mathfrak{N}\mathbf{H}_2)\cdot(\mathfrak{M}\mathbf{E}_2)\\ + 4\mathbf{E}_1\cdot(\mathfrak{N}\cdot\mathbf{E}_2) - 4\mathbf{E}_1\cdot(\mathfrak{M}\mathbf{H}_2) + 4\mathbf{H}_1\cdot(\mathfrak{N}\mathbf{H}_2) + 4\mathbf{H}_1\cdot(\mathfrak{M}\mathbf{E}_2).\end{aligned} \tag{11}$$

The analogy between this classical effect and the quantum interference effect for charged particles moving around a region enclosing electromagnetic flux is evident. We have here a shift in the interference pattern which is nonzero only if the light wave moves in the whole multiply-connected region, while in any simply connected region free of curvature the effect vanishes.

As a final sample, consider two twins in two space ships initially situated at point 1 of Fig. 1. Their clocks are synchronized and they agree to follow orbits C_R and C_L, by accelerating themselves in equal amounts with respect to their own frame of reference at equal proper times.

Assume first that the motion takes place in an overall flat space–time. Then, clearly, when they meet again at point 2 they are still of the same age.

Suppose now that the orbit made by the two space ships encircles a bounded region of curved space–time. Then, in general, the metric tensor will not be Minkowskian everywhere along the path of both space ships. In that case the twins will differ in their age when they meet at point 2 by an amount

$$\Delta s = \int_{CL} ds - \int_{CR} ds. \tag{12}$$

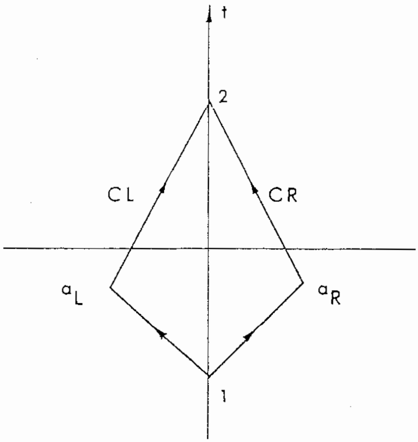

FIG. 1

That this result is due to a nonlocal effect of the enclosed curvature is evident from the fact that all experiments that can be carried out by the twins inside their rockets will give results identical to the case with no curvature present anywhere. Each of the twins can set up a nonsingular coordinate system in such a way that the metric tensor will be Minkowskian in a simply-connected region containing his orbit. On the other hand, an external observer who wants to compare the age of the twins at all intermediate stages of the trip, has to set up a coordinate system including both orbits. This cannot be done with a single non-singular coordinate frame covering the non-simply-connected region.

III

In this section we discuss the Yang–Mills field and its interaction with nucleons. As it is well known (*3*) this gauge field was introduced to account for the isospin symmetry of strong interactions. The equations of motion for a nucleon interacting with the Yang–Mills field are

$$\gamma^\mu \left(\partial_\mu - \frac{ig}{\hbar c} \tau \cdot B_\mu \right) \Psi + m\Psi = 0, \tag{13}$$

where τ^i are the Pauli matrices, g the coupling constant, and $B_\mu{}^i$ the three-component Yang–Mills potential. The field derived from this potential is

$$F_{\mu\nu}^i = B_{\nu,\mu}^i - B_{\mu,\nu}^i + ige_{jk}^i B_\mu{}^j B_\nu{}^k, \tag{14}$$

where e_{jk}^i is the totally antisymmetric Levi–Civita tensor.

Consider a case in which the field $F_{\mu\nu}^i$ is different from zero in a bounded space–

time region R. In analogy with our previous examples we will investigate the nonlocal effects of $F^i_{\mu\nu}$ on the motion of a nucleon around R. For simplicity we discuss a case in which the field has only one component in the direction of τ^1. The wavefunction of a nucleon after moving around R is given by

$$\Psi_f' = \exp(-i\tau'\Phi)\Psi_f, \tag{15}$$

where

$$\Phi = \frac{g}{\hbar c}\oint B^1_\mu\, dx^\mu = \frac{g}{\hbar c}\iint F^1_{\mu\nu}\, ds^{\mu\nu}$$

and Ψ_f is the wavefunction that would result for the same motion for $F^i_{\mu\nu} = 0$ everywhere.

From (15) we get

$$\Psi_f' = [\cos\Phi - i\tau^1 \sin\Phi]\Psi_f. \tag{16}$$

If the initial state Ψ_i represents a proton

$$\tau^3\Psi_i = \Psi_i, \tag{17}$$

then Ψ_f' (the state after the motion on the close orbit surrounding R) will also correspond to a proton. But when $F^i_{\mu\nu}$ is different from zero we have Ψ_f corresponding to a superposition of proton and neutron states with probabilities $\cos^2\Phi$ and $\sin^2\Phi$.

The above description of the effect is not yet fully gauge-invariant since we singled out the Z axis in isospace. To obtain an effect that is completely gauge-invariant, consider the following experiment. Two nucleons are initially located at the same point in space and have the same charge. One of them is moved around R, the other stays behind. When the two meet again the relative orientation of their isospin is measured (by comparing their charge). The result depends on the enclosed flux and is obviously gauge-invariant.

As in the gravitational case there is no gauge invariant way to trace the change in the charge of the nucleon.[2] Moreover, if the motion of the nucleon is confined to a simply-connected region outside R, no effect of the flux will be observed. An interesting open question is whether nonlocal experiments of this kind are feasible. Such experiments might provide a direct check on the existence of Yang–Mills fields.

IV

The previous discussion makes it clear that nonlocal effects of potentials exist in classical as well as in quantum theories. However, there still seems to exist an important difference between the two cases.

[2] Nevertheless, in any gauge the total charge (field plus nucleon) is conserved and the change in the charge of the nucleon is compensated by the change in the charge of the Yang–Mills field.

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Compare the cases of a classical vector field transported around a bounded region of curved space–time, and a quantum charged particle traveling around an electromagnetic flux. In the first case the direction of the vector will change along the orbit, the change being a Lorentz transformation. At each point in the orbit the direction of the vector relative to its initial direction is not gauge-invariant. Still, different observers can set up their own coordinate systems, with respect to which the direction can be observed locally. In general, these frames will be nonintegrable globally, but nevertheless suitable for any simply connected region outside the flux (for instance, the different cones in the gravitational example in II). Different observers will, in general, disagree about the local changes measured, but the net result at the end of the orbit will be the same for all observers.

In the quantum case the phase of the wavefunction will also change locally in a non-gauge-invariant way, but here a further restriction seems to exist: there is no way to set up even a non-gauge-invariant frame relative to which the local phase can be observed. However, this further restriction is not a common characteristic of all quantum effects of potentials. Indeed even for electromagnetic interactions there are quantum examples for which the analogy to the classical case is complete.

Consider a charged-meson field described by a complex field operator Ψ, which can be written in the form

$$\Psi = R(x, t)e^{i\phi(x,t)}, \tag{18}$$

where R and ϕ are real operators which commute at space like points. The usual commutation relations imply that the charge density

$$\rho = -ie(\Psi^*\dot{\Psi} - \dot{\Psi}^*\Psi) \tag{19}$$

is conjugate to the phase operator ϕ modulo 2π; that is,

$$[\rho(x, t); e^{i\phi(x',t)}] = ie^{i\phi(x,t)}\delta^3(x - x'). \tag{20}$$

It can be shown (*8*) that the meson field can be prepared in a state which is a coherent superposition of eigenstates of charge with definite relative phase. A non-gauge-invariant frame of reference can be set up relative to which the phase of Ψ can be observed locally.

When a packet of field Ψ is transported around a circuit enclosing electromagnetic flux, it will pick up a phase proportional to the line integral of the potential between the starting point and the point in question. An array of apparatus is set up along the trajectory which can measure the phase of the field locally at each state of the process. Thus, the analogy to the gravitational field is complete.

V

Nonlocal effects of the type we have considered are not necessarily a consequence of the existence of potentials in the formulation of the theory. Indeed it is possible to find equivalent local theories in which only gauge-invariant quantities are used, and where nevertheless similar effects occur.

Let us start with the case of a nonrelativistic quantum particle interacting with an electromagnetic field. The action expressed in terms of the vector potential $\mathbf{A}$, the scalar potential Φ and the wavefunction Ψ, is

$$S = \int d^3x\,dt \left\{ \frac{1}{2m}\,\hbar \left(\boldsymbol{\nabla} + \frac{ie}{\hbar c}\,\mathbf{A} \right) \Psi^* \hbar \left(\boldsymbol{\Delta} - \frac{ie}{\hbar e}\,\mathbf{A} \right) \Psi \right.$$
$$\left. - \Psi^* \left(i\hbar \frac{\partial}{\partial t} - \frac{e}{c}\,\Phi \right) \Psi \right\} + S_{\mathrm{e.m.}}, \tag{21}$$

where $S_{\mathrm{e.m.}}$ is the action for the free electromagnetic field. We write

$$\Psi = Re^{i\phi}, \qquad \Psi^* = Re^{-i\phi}, \tag{22}$$

where R and ϕ are real. We then get

$$S = \int d^3x\,dt \left\{ \frac{1}{2m}\,\hbar \left[\boldsymbol{\nabla} R + iR \left(\frac{e}{\hbar c}\,\mathbf{A} - \boldsymbol{\nabla}\phi \right) \right] \hbar \left[\boldsymbol{\nabla} R - iR \left(\frac{e}{\hbar c}\,\mathbf{A} - \boldsymbol{\nabla}\phi \right) \right] \right.$$
$$\left. - i\hbar R\,\frac{\partial R}{\partial t} + \hbar\,R^2 \left(\frac{e}{\hbar c}\,\Phi + \frac{\partial \phi}{\partial t} \right) \right\} + S_{\mathrm{e.m.}}\,. \tag{23}$$

The equations of motion obtained by varying S with respect to ϕ, R, $\mathbf{A}$, and Φ are

$$\begin{gathered} \frac{\hbar}{2}\frac{\partial}{\partial t} R^2 + \frac{\hbar}{2m}\,\boldsymbol{\nabla} \cdot \left[R^2 \left(\boldsymbol{\nabla}\phi - \frac{e}{\hbar c}\,\mathbf{A} \right) \right] = 0, \\ \hbar\,R\,\frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2m}\,R \left(\boldsymbol{\nabla}\phi - \frac{e}{\hbar c}\,\mathbf{A} \right)^2 = \frac{1}{2m}\,\hbar^2 \nabla^2 R - \frac{e}{c}\,R\Phi, \\ \nabla^2\,\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\,R^2\,\frac{\hbar e}{mc} \left(\boldsymbol{\nabla}\phi - \frac{e}{\hbar c}\,\mathbf{A} \right), \\ \nabla^2\,\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\,\frac{e}{c}\,R^2\,. \end{gathered} \tag{24}$$

If we call

$$\rho = R^2; \qquad \mathbf{V} = \frac{\hbar}{m} \left(\boldsymbol{\nabla}\phi - \frac{e}{\hbar c}\,\mathbf{A} \right); \qquad j = \frac{e}{c}\,\rho\mathbf{V};$$
$$q = \frac{e}{c}\,\rho, \quad \text{and} \quad \bar{\Phi} = \frac{\hbar}{m}\frac{\partial \phi}{\partial t} + \frac{e}{mc}\,\Phi,$$

we can write (24) in terms of gauge-invariant local quantities in the form

$$\partial\rho/\partial t + \boldsymbol{\nabla}\cdot(\rho\mathbf{V}) = 0,$$

$$\bar{\Phi} + \frac{1}{2} V^2 = \frac{\hbar^2}{2m^2} \frac{\nabla^2 R}{R},$$

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot V) + \frac{1}{c^2} \boldsymbol{\nabla} \frac{\partial}{\partial t} \bar{\Phi} = \frac{e}{mc}\, \mathrm{j}. \tag{25}$$

$$\nabla^2 \bar{\Phi} - \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot V = -\frac{e}{mc}\, q.$$

These equations can be interpreted as the continuity equation, Bernoulli's equation, and Maxwell's equation for a charged fluid with density ρ and quantum pressure $P = (\hbar^2/2m^2)(\nabla^2 R/R)$ [see Ref. (*9*)]. The observable quantities of the fluid at a point are ρ and $\mathrm{j} = (e/c)\rho V$, thus the velocity is defined only in the regions where $\rho \neq 0$.[3]

Consider now a situation in which the field flows around an impenetrable body enclosing a magnetic field. When passing the body, the fluid will split into two separate streams. When these streams are re-combined later, an interference pattern of their charge density will be produced. As long as the two streams do not overlap the observable quantities ρ and J will remain independent of the magnetic flux. However, when they finally meet the charge distribution ρ will depend on the enclosed flux in the manner shown in Ref. (*1*).

The equation of motion for ρ is local (and, therefore, independent of the magnetic flux which is far away) and we would expect that if all the local quantities are independent of the flux before the streams overlap, then they should remain so. Thus Eqs. (25) seem to be ambiguous.

Suppose that the body enclosing the flux is not completely impenetrable, so that ρ inside the body (ρ_{in}) will be very small but not exactly zero. The magnetic field will now affect the fluid even before the interference takes place, and it is possible now to trace the effect of the flux locally. Yet the local effects of the magnetic field are proportional to ρ and therefore extremely small but its effect on the interference pattern is always finite (even when $\rho_{\mathrm{in}} \to 0$). If we compare two situations in which the enclosed magnetic flux is different we see that before the overlap takes place the difference in ρ and j in the two cases is arbitrarily small, and, nevertheless, after the two streams meet the difference is finite. In this sense Eqs. (25) might be said to include instabilities. These instabilities are removed when the non-gauge-invariant quantities are introduced. Thus in this context we may interpret the electromagnetic potentials as quantities useful in removing instabilities.[4]

[3] In the regions where $\rho = 0$, Eq. (26) do not determine V.

[4] Note that the condition of single valuedness of the wavefunction implies that along any closed loop $\oint V.dl = (h/m)$ integer $-(e/mc)$. Flux included. Since this constraint involves the velocity but not the current it does not affect our conclusion concerning the inestability of ρ and j.

The above treatment may be extended to general gauge theories. Consider again the interaction of a general gauge field with a scalar field ϕ. We split the field ϕ in a way which corresponds to the phase and absolute value of the Schrödinger wavefunction

$$\phi = D(\epsilon)\phi_0 = \exp[\epsilon^i(x)T_i]\phi_0(x), \tag{26}$$

where $\epsilon^i(x)$ and $\phi_0(x)$ are scalars and T_i are the generators of the group in the representation of ϕ. Using the field ϕ and its contragradient representation

$$\phi^{\sim} = D^{-1\sim}\phi_0 \; (D_{ij}^{\sim} = D_{ji}) \tag{27}$$

we form the scalar Lagrangian density which describes our interaction

$$L = [\partial_\mu(D^{-1\sim}\phi_0) - \alpha B^i{}_\mu T_i^{\sim} D^{-1\sim}\phi_0] \cdot [\partial_\mu(D\phi_0) + \alpha B^i{}_\mu T_i D\phi_0] + m^2\phi_0^2 + L_{\text{gauge}} . \tag{28}$$

where $T_i^{\sim}$ are the generators corresponding to the contragradient representation.

Using the relations [see Ref. (4), p. 671],

$$\partial_\mu D = \partial_\mu \epsilon^a R_a^{-1b} D T_b ,$$
$$\partial_\mu D^{\sim -1} = -\partial_\mu \epsilon^a R^{-1b}{}_a D^{-1\sim} T_b^{\sim} ,$$

we have

$$\begin{aligned} L = {} & [-\partial_\mu\epsilon^a R^{-1b}{}_a D^{-1\sim} T_b^{\sim}\phi_0 + D^{-1\sim}\partial_\mu\phi_0 - \alpha B^a{}_\mu T_a^{\sim} D^{-1\sim}\phi_0] \\ & \times [\partial_\mu\epsilon^a R^{-1b}{}_a D T_b\phi_0 + D\partial_\mu\phi_0 + \alpha B^a{}_\mu T_a D\phi_0] + m^2\phi_0^2 + L_B \\ = {} & \partial_\mu\phi_0\partial_\mu\phi_0 + \partial_\mu\epsilon^a R^{-1b}{}_a(\partial_\mu\phi_0 T_b\phi_0 - \phi_0 T_b\partial_\mu\phi_0) \\ & + \alpha B^a{}_\mu(\partial_\mu\phi_0 D^{-1}T_a D\phi_0 - \phi_0 D^{-1}T_a D\partial_\mu\phi_0) \\ & - \partial_\mu\epsilon^a R^{-1b}{}_a\phi_0 T_b \partial_\mu\epsilon^{a'} R_{a'}^{-1b'} T_b{}'\phi_0 - \alpha^2 B^a{}_\mu B^{a'}{}_\mu\phi_0 D^{-1}T_a T_a{}' D\phi_0 \\ & - \alpha\partial_\mu\epsilon^a R^{-1b}{}_a B^{a'}{}_\mu[\phi_0 D^{-1}T_a{}' D T_b\phi_0 + \phi_0 T_b D^{-1} T_a{}' D\phi_0] \\ & + m^2\phi_0^2 + L_B . \end{aligned} \tag{30}$$

From the commutation relation between D and T_a,

$$[T_a , D] = -R^{-1}{}_{ac}C^c_{ed}\epsilon^e D T_d , \tag{31}$$

we deduce

$$D^{-1}T_a D = (\delta_{ad} - R^{-1}{}_{ac}C^c{}_{ed}\epsilon^e)T_d \tag{32}$$

and

$$\tfrac{1}{2}D^{-1}(T_a T_a{}' + T_a' T_a)D = \tfrac{1}{2}(\delta_{ad} - R^{-1}{}_{ac'}C^{c'}{}_{e'd}\epsilon^{e'})(\delta_{a'd'} - R^{-1}{}_{a'c}C^c{}_{ed'}\epsilon^e) \cdot (T_d T_{d'} + T_{d'} T_d). \tag{33}$$

Using (32) and (33) the Lagrangian (30) may be cast into the form

$$\begin{aligned} L = {} & \partial_\mu\phi_0\partial_\mu\phi_0 + [\partial_\mu\epsilon^a R^{-1b}{}_a + \alpha B^b{}_\mu - \alpha B^a{}_\mu R^{-1}{}_{ac} C^{c}{}_{e}{}^{b} \epsilon^e] \\ & \times [\partial_\mu\phi_0 T_b\phi_0 - \phi_0 T_b\partial_\mu\phi_0] \\ & - [\partial_\mu\epsilon^a R^{-1b}{}_a + \alpha B^b{}_\mu - B^a{}_\mu R^{-1}{}_{ac} C^{c}{}_{e}{}^{b}\epsilon^e] \\ & \times [\partial_\mu\epsilon^{a'} R^{-1b'}{}_{a'} + \alpha B^{a'}{}_\mu - B^{a'}{}_\mu R^{-1}{}_{a'c'} C^{c}{}_{e'b'}\epsilon^{e'}]\phi_0 T_b T_{b'}\phi_0 \\ & + m^2\phi_0^2 + L_B\,. \end{aligned} \tag{34}$$

ϕ_0 may be written as

$$\phi_0 = f(x)\cdot u \tag{35}$$

where $f(x)$ is a scalar and u a constant representation of the group independent of x. We then have $\phi = f(x)e^{\epsilon(x)\cdot T}u$, i.e., ϕ is obtained at each point by performing a space dependent "rotation" on the fixed u and changing its length by a scalar factor $f(x)$.

Furthermore we choose u to satisfy

$$u(T_aT_b + T_bT_a)u = u^2\gamma_{ab}\,, \tag{36}$$

where

$$\gamma_{ab} = C^e{}_{af}C^f{}_{be}\,. \tag{37}$$

We now have

$$(\partial_\mu\phi_0)T_a\phi_0 - \phi_0 T_a\partial_\mu\phi_0 = 0, \tag{38}$$

$$\phi_0(T_aT_b + T_bT_a)\phi_0 = f^2u^2\gamma_{ab} \tag{39}$$

and, using the matrix γ to rise and lower indices and the relation [see (4), p. 666, Eqs. (11.43)]

$$R^{-1}{}_{ac}C^{c}{}_{e}{}^{b}\epsilon^e = \delta_a{}^b - D^{(A)-1b}{}_a\,, \tag{40}$$

we can write

$$\begin{aligned} L = {} & \partial_\mu f\partial_\mu f + m^2f^2u^2 - [\partial_\mu\epsilon^a R^{-1b}{}_a + \alpha D^{(A)-1b}{}_a B^a{}_\mu] \\ & \qquad \times [\partial_\mu\epsilon^{a'} R^{-1}{}_{ba'} + \alpha D^{(A)-1}{}_{ba'}B^{a'}{}_\mu]f^2u^2 + L_B \end{aligned} \tag{41}$$

The equations of motion for f and ϵ^b following from (41) are

$$\Box f + (V_\mu{}^b V_{\mu b} - m^2)f = 0 \tag{42}$$

and

$$\partial_\mu(V^b{}_\mu f^2) = 0, \tag{43}$$

where

$$V_\mu{}^b = \partial_\mu\epsilon^a R^{-1b}{}_a + \alpha D^{(A)}{}_{(\epsilon)}{}^{-1b}{}_a B^a{}_\mu \tag{44}$$

Equations (42) and (43) are the generalization of equations (25) where the quantities f^2 and $V^b{}_\mu$ play the role of density and velocity of the fluid and can be shown to be gauge invariant (see Appendix I). In this case we have again that the observables of the fluid are f^2 and $j^b{}_\mu = f^2 V^b{}_\mu$. The velocity $V^b{}_\mu$ is defined and observable in the regions where $f^2 \neq 0$. An argument similar to that given in the discussion of the special case of the quantum fluid connected with the non-relativistic Schrödinger equation may be reproduced here. In a situation corresponding to interference in non-simply-connected regions, Eqs. (42) and (43) will develop instabilities analogous to the ones described previously.

To complete the discussion we now show that the equations for the gauge field can be written in terms of local gauge-invariant quantities.

Under a gauge transformation,

$$\begin{aligned} F^b{}_{\mu\nu} &\longrightarrow F'^b{}_{\mu\nu} = D^{(A)b}{}_d F^d{}_{\mu\nu}\,, \\ F_{b\mu\nu} &\longrightarrow F'_{b\mu\nu} = D^{(A)-1e}{}_b F_{e\mu\nu}\,, \end{aligned} \tag{45}$$

and $L_B = (1/2\alpha) F^b{}_{\mu\nu} F_{b\mu\nu}$ is gauge-invariant.

The variation of the Lagrangian density (41) leads to the equations of motion

$$-\alpha u^2 f^2 D^{(A)}{}_{(\epsilon)}{}^{-1b}{}_k V_{b\mu} = \frac{1}{\alpha} F_{k\mu\nu;\nu} \tag{46}$$

or

$$-\alpha^2 u^2 f^2 V_{k\mu} = D^{(A)}{}_{(\epsilon)}{}^b{}_k F_{b\mu\nu;\nu} = F^0{}_{k\mu\nu;\nu} \tag{47}$$

where $F^0{}_{b\mu\nu}$ is the field obtained from $F_{b\mu\nu}$ by a gauge transformation with parameter $-\epsilon$. In that gauge we have from (44)

$$V^b{}_\mu = B^b{}_\mu$$

so that

$$F^0{}_{b\mu\nu} = V_{b\nu,\mu} - V_{b\mu,\nu} + C_{bef} V^e{}_\mu V^f{}_\nu\,. \tag{48}$$

Equation (47) can be written in terms of gauge-invariant quantities in the form

$$-\alpha^2 u^2 f^2 V^k{}_\mu = F^{0k}{}_{\mu\nu,\nu} + C^k{}_{em} V^e{}_\nu F^{0m}{}_{\mu\nu} \tag{49}$$

It is then possible to show (see Appendix II) that Eq. (43) follows as a consequence of the field equations (49).

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APPENDIX I

It is clear from the definition that f is gauge-invariant. Let us now show that $V^i{}_\mu$ is also gauge-invariant.

Suppose that $\epsilon = F(a, b)$, then

$$\begin{aligned} V'^{i}_{\mu} &= \partial_\mu \epsilon^j R^{-1i}{}_j(\epsilon) + D^{(A)}{}_{(\epsilon)}{}^{-1i}{}_k \alpha B'^k{}_\mu \\ &= \partial_\mu F^j(a, b) R^{-1i}{}_j(F) + D^{(A)-1i}{}_k(F)[-L^{-lk}{}_n(a) a^n{}_{,\mu} \\ &\quad + \alpha D^{(A)k}{}_n(a) B^n{}_\mu] \text{ [see Ref (4), p. 685, Eqs. (12.14)]} \\ &= \frac{\partial F^j}{\partial a^n}(a, b) a^n{}_{,\mu} R^{-1i}{}_j(F) + \frac{\partial F^j}{\partial b^n}(a, b) b^n{}_{,\mu} R^{-1i}{}_j(F) \\ &\quad - D^{(A)-1i}{}_k(F) L^k{}_n(a) a^n{}_{,\mu} + \alpha D^{(A)-1i}{}_k(F) D^{(A)k}{}_n(a) B^n_\mu , \end{aligned} \tag{A1}$$

Using the relations [see (4), p. 663, Eqs. (11.17) and (11.18)]

$$(\partial F^j/\partial a^n)(a, b) = L^j{}_k(F) L^{-1k}{}_n(a) \tag{A2}$$

and

$$(\partial F^j/\partial b^n)(a, b) = R^j{}_k(F) R^{-1k}{}_n(b), \tag{A3}$$

we can write (A2) in the form

$$\begin{aligned} V'^{i}_{\mu} = {} & L^j{}_k(F) L^{-1k}{}_n(a) a^n{}_{,\mu} R^{-1i}{}_j(F) + R^j{}_k(F) R^{-1k}{}_n(b) b^n{}_{,\mu} R^{-1i}{}_j(F) \\ & - D^{(A)-1i}{}_k(F) L^k{}_n(a) a^n{}_{,\mu} + \alpha D^{(A)-1i}{}_k(F) D^{(A)k}{}_n(a) B^n{}_\mu . \end{aligned} \tag{A4}$$

The first and third terms in (A4) cancelled out by virtue of the relation

$$L^j{}_k(F) R^{-1i}{}_j(F) = D^{(A)-1i}{}_k(F) \tag{A5}$$

and

$$D^{(A)-1}(F) D^{(A)}(a) = D^{(A)-1}(b), \tag{A6}$$

so finally we can write (A4) in the form

$$V^{i'}{}_\mu = R^{-1i}{}_n(b) b^n{}_{,\mu} + \alpha D^{(A)-1i}{}_k(b) B^k{}_\mu = V_\mu{}^i .$$

APPENDIX II

We take the divergence on both sides of Eq. (49) and get

$$-\alpha^2 u^2 \partial_\mu(f^2 V^k{}_\mu) = F^{0k}{}_{\mu\nu,\nu\mu} + C^k{}_{em} V^e{}_\nu F^{0m}{}_{\mu\nu,\mu} + C^k{}_{em} V^e{}_{\nu,\mu} F^{0m}{}_{\mu\nu} . \tag{A7}$$

The first term on the right hand side of (A7) vanishes by virtue of the antisymmetry of $F^{0m}{}_{\mu\nu}$. Again using Eq. (49) and the relation $C^k{}_{em} = -C^k{}_{me}$, we write (A7) in the form

$$\begin{aligned} -\alpha^2 u^2 \partial_\mu(f^2 V^k{}_\mu) &= C^k{}_{em} V^e{}_{\nu,\mu} F^{0m}{}_{\mu\nu} - C^k{}_{em} C^m{}_{jn} V^j{}_\mu F^{0n}{}_{\mu\nu} V^e{}_\nu \\ &= \tfrac{1}{2} C^k{}_{em}(V^e{}_{\nu,\mu} - V^e{}_{\mu,\nu}) F^{0m}{}_{\mu\nu} - C^k{}_{em} C^m{}_{jn} V^j{}_\mu V^e{}_\nu F^{0n}{}_{\mu\nu} . \end{aligned} \tag{A8}$$

From the Jacoby identity

$$C^k{}_{em} C^m{}_{jn} + C^k{}_{jm} C^m{}_{ne} + C^k{}_{nm} C^m{}_{ej} = 0, \tag{A9}$$

we deduce

$$C^k{}_{em}C^m{}_{jn}V^j{}_\mu V^e{}_\nu F^{0n}{}_{\mu\nu} = -\tfrac{1}{2}C^k{}_{nm}C^m{}_{ej}V^j{}_\mu V^e{}_\nu F^{0n}{}_{\mu\nu} \qquad (A10)$$

Using (A10) we can write (A8), after changing indices, in the form

$$-\alpha^2 u^2 \partial_\mu(f^2 V^k{}_\mu) = \tfrac{1}{2}C^k{}_{em}F^{0m}{}_{\mu\nu}[V^e{}_{\nu,\mu} - V^e{}_{\mu,\nu} + C^e{}_{jn}V^j{}_\mu V^n{}_\nu]$$
$$= \tfrac{1}{2}C^k{}_{em}F^{0m}{}_{\mu\nu}F^{0e}{}_{\mu\nu} = 0.$$

We finally remark that the system of Eqs. (42), (49) can be derived from an alternative Langrangian density, namely

$$L = [\partial_\mu f \partial_\mu f + m^2 f^2 - V_\mu{}^b V_{b\mu} f^2]u^2 + \frac{1}{2\alpha^2} F^{ob}{}_{\mu\nu} F^o{}_{b\nu\mu}.$$

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