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Modular Variables in Quantum Theory†

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Abstract

The non-local aspects of interaction between quantum systems are investigated. These aspects are particularly conspicuous in quantum phenomena without a classical analog, such as the quantum effects of electromagnetic potentials. The study of the potential effect leads to the introduction of a new type of dynamical variable, the modular variable, which brings out the physical features of quantum mechanical non-locality.

1. *Introduction*

In the attempts to understand the basic aspects of quantum mechanics the study of quantum effects without a classical analog has played a particularly important role. Among such quantum effects the most familiar is the interference of material particles, such as electrons. As is well known, the analysis of the two-slit electron interference experiment has provided much insight into the physical meaning of the indeterminacy relations and has led to an improved terminology for the description of quantum phenomena (Bohr, 1949). In recent years the study of quantum effects without a classical analog has acquired renewed interest as a result of the discovery of the quantum effects of electromagnetic potentials (Aharonov & Bohm, 1959). This discovery not only revealed a whole set of interesting

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phenomena that had hitherto been overlooked, but it also placed the question of the physical description of quantum behavior in a new light. In particular, it suggested a new approach to the problem of non-locality in the quantum domain.

The following is an account of some of the work on the non-locality problem that has been inspired by the potential effect. The principal result of this work is the emergence of a new type of dynamical variable that seems to be the physical expression of the features of non-locality. It looks as if this variable, which we call the modular variable, is the key element of the physical description of quantum phenomena which have no classical analog.

In Section 2 we review the potential effect. In Section 3 we show that this effect can be described as an exchange of an otherwise conserved dynamical quantity, namely modular momentum. In Section 4 we investigate some general properties of modular variables. In particular, we demonstrate that for these variables the quantum equations of motion are essentially different from the classical ones. In Section 5 we introduce and examine the concept of modular energy. Section 6 contains some concluding remarks.

2. *The Potential Effect*

The discovery of the potential effect resulted from an attempt to investigate the observability of potentials in the quantum domain. In order to get acquainted with the background of this question, let us first consider the problem of potentials in classical physics. For predicting the orbit of a particle it is sufficient to know the initial conditions and the forces that act on the particle and determine its acceleration. What, then, is the role of potentials?

The most familiar example is the electromagnetic case, where the potentials are used mainly as mathematical auxiliaries to express the field equations in a canonical form. The connection between the field quantities **E**, **B** and the potentials φ, **A** is given by the equations

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Obviously, even after the values of **E** and **B** have been specified, φ and **A** are still to some extent arbitrary. Different values of φ and **A** may correspond to the same physical situation: if we perform a so-called gauge transformation

$$\varphi \to \varphi - \frac{1}{c}\frac{\partial \chi}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\chi$$

where χ is an arbitrary function of space and time, the forces and thus the accelerations will not be affected. Since therefore the orbits are left unchanged, such a gauge transformation will give rise to no observable effects in classical theory.

However, once the potentials are introduced we are faced with a curious situation. To see this, let us consider the following set-up: let R be a cylindrical region containing a magnetic field $\mathbf{B}$, while outside this region both $\mathbf{B}$ and $\mathbf{E}$ are zero. We confine our attention to the field-free region S outside R. Any classical experiment performed in S will yield exactly the same results as the corresponding experiment performed in the vacuum. Yet, it is impossible to find a gauge in which φ and $\mathbf{A}$ are zero everywhere in S. This is due to the fact that the region S is not simply connected. In other words, there are circles in S which cannot be shrunk to a point without penetrating R. Along any such circle, according to Stoke's theorem,

$$\oint \mathbf{A}.d\mathbf{l} = \int (\nabla \times \mathbf{A}).d\mathbf{a} = \int \mathbf{B}.d\mathbf{a} \neq 0$$

Thus, there exists in S a gauge invariant quantity, $\oint \mathbf{A}.d\mathbf{l}$, which is unobservable, since no classical experiment performed within the region can distinguish between the vacuum case where this quantity is zero and our case where it is different from zero. This peculiar state of affairs might have led a sceptical classical physicist to question the wisdom of introducing the electromagnetic potentials, since they lead to an unavoidable inequivalence of the mathematical description and the physics of multiply connected regions.

We shall now see that in the quantum domain this discrepancy between the mathematics and the physics disappears. Indeed, in this domain there corresponds to every gauge invariant mathematical quantity a physical experiment that could measure its value. As a simple example, let us consider a situation involving only a scalar potential φ. Suppose a charged particle is placed in a region of space where the scalar potential is only a function of time. A practical way to do this is to enclose the particle in a Faraday cage inside which φ is made time dependent by changing the amount of charge on the surface of the cage.

Let us compare the mathematical description of this set-up in the classical and the quantum case. Classically, the behavior of the enclosed particle is governed by the Lorentz force

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

Because of the space independence of φ both $\mathbf{E}$ and $\mathbf{B}$ are zero. Varying the charge on the surface of the cage thus has no effect at all on the motion of the particle.

Quantum mechanically, the situation at first sight appears different, since the basic equation contains the potentials rather than the fields. Indeed, the Schrödinger equation for the case considered is

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \qquad\qquad H = H_0 + e\varphi,$$

where H_0 is the Hamiltonian describing the motion without the time-dependent potential $\varphi(t)$. Thus, if ψ_0 is the wave function describing the state of the particle before φ is turned on, and ψ describes the state after φ is turned on, ψ cannot equal ψ_0. It is easy to verify by substitution that $\psi = \psi_0 \exp(-i\alpha)$, where

$$\alpha = \frac{e}{\hbar} \int_{t_0}^{t} \varphi(t)\,dt.$$

Nevertheless, this difference between ψ and ψ_0 is of no consequence in the present case. As is well known, ψ itself is unobservable; only bilinear combinations of ψ and ψ^* have physical significance, and in such combinations the purely time-dependent phase disappears. For a long time this led to the belief that in quantum mechanics the potentials have no direct physical significance.

The problem under discussion is a good testing ground for our way of thinking about quantum phenomena. As far as classical physics goes, the above analysis has exhausted the possibilities of experimenting with a charged particle in a field-free region. If we take the customary view that the possibilities of measurement are in general more restricted in the quantum domain than in the classical domain, we would be inclined to think that the analysis has also exhausted the possible experiments in quantum physics. If, however, we regard the quantum description as of greater richness than the classical scheme, we would be less convinced that this was the case. We might then try to exploit the new potentialities of experimentation, i.e. those quantum experiments that have no classical analog, for the purpose of demonstrating physical effects of the scalar potential.

Actually we need not look very far. Consider two spatially separated Faraday cages. Clasically, the charged particle may be in one cage or the other, but not in both together. Quantum mechanically, however, there is a third possibility: the particle may be 'shared' simultaneously by both of the field-free regions. This corresponds to describing the state of the particle as a superposition of two wave packets, $\psi_1 + \psi_2$, where ψ_1 is different from zero only inside the first Faraday cage and ψ_2 is different from zero only inside the second Faraday cage. Let us exploit this new physical possibility in the following way. In the first cage we produce a time-dependent potential; in the second cage we keep the potential fixed. The state of the charged particle is now $\psi_1 \exp(-i\alpha) + \psi_2$, where again

$$\alpha = \frac{e}{\hbar} \int_{t_0}^{t} \varphi(t)\,dt.$$

We see that in this case the difference between the state with the altered potential and the original state does not correspond to a multiplicative phase. The two states are therefore physically inequivalent. Indeed, if we open the cages and bring the two wave packets to interfere with each other, the resulting interference pattern will depend on the relative phase α between the two packets and thus on the difference of potential between the two cages.

There is a similar observable effect of the vector potential. If a charged particle passes through a region in which there is a vector potential **A**, the phase of its wave function increases by $e/\hbar c$ multiplied by the integral of the vector potential along the trajectory, $\int \mathbf{A}.d\mathbf{l}$. To make this phase change observable we consider an interference experiment around a shielded flux region. The flux produces a relative phase change of the wave packets passing on either side, which is proportional to the circulation of the vector potential outside the flux region. This phase change shifts the whole interference pattern.

The idea that electromagnetic potentials may have observable effects in field-free regions gave rise to a stir among physicists. Apart from the natural surprise that it was still possible to uncover new general aspects of a topic that had been so thoroughly analyzed for more than thirty years, the initial disbelief arose from two sources. In the first place, there was a strong suspicion that the effects could not be consistent with the principles of quantum theory. In the second place, experimental data were available that seemed incompatible with the predicted effects.

The consistency question was treated by Furry & Ramsey (1960), who showed that the effects of potentials, far from contradicting the principles of quantum theory, are necessary for consistency of that theory. If these effects did not exist, it would be possible, by measuring the potentials induced by an electron in a detecting device, to ascertain its path through an interferometer without concomitant destruction of the interference pattern.

The belief that the potential effects were incompatible with experimental data arose when Marton reported that when he performed his electron-interference experiments (Marton, *et al.*, 1954) stray magnetic field had been present in his apparatus. If the potential effects existed, these fields apparently should have shifted the interference pattern about 1000 fringes 60 times per second, and under these circumstances it was hard to understand how interference could have been observed at all. However, a closer analysis of Marton's set-up (Werner & Brill, 1960) showed that besides the phase change due to the vector potential, account had to be taken of the beam-bending effect of a magnetic field, and it turned out that the two magnetic effects almost completely compensated each other. It thus became clear that Marton's experiment should be considered an indirect confirmation of the predicted potential effects.

In the last few years, several experiments have been made to check the quantum effects of the potentials (Chambers, 1960; Fowler, *et al.*, 1961; Boersch, *et al.*, 1960, 1961a, b, 1962a, b; Möllenstedt & Bayh, 1962a, b; Jaklevic *et al.*, 1964a, b). These experiments have so far been confined to the magnetic case; they have given a clear demonstration of the existence of the effect.

3. *The Grating-solenoid Paradox and Modular Momentum*

The experiments that exhibit the potential effect show that although the field vanishes in the region accessible to the charged particle there is, nevertheless, some sort of interaction between the particle and the source. In classical physics an interaction can be defined as an exchange of an otherwise conserved physical quantity between the two interacting systems. What we mean by saying that classically the source of potential does not interact with a charged object moving in field-free regions, is that no momentum is exchanged between the source and the object. We may now ask: is it possible in the quantum case to consider the potential effect as an exchange of some conserved physical quantity between the charged particle and the source of potential?

Consider again the example of a particle being shared by two separate regions of space. To begin with, the quantum state is $\psi_0 = \psi_1 + \psi_2$. Then, by means of the two Faraday cages and a battery, a relative phase is introduced between the two wave packets, resulting in a state $\psi_\alpha = \psi_1 + \exp(i\alpha)\psi_2$. Our question is: do there exist conserved quantities that have different values in ψ_0 and ψ_α?

In view of the Ehrenfest theorem, according to which the equation of motion for the average of a quantum observable is similar to the equation of motion for the corresponding classical observable, one might argue: since classically the equations of motion in the multiply connected field-free region are exactly the same as for the vacuum case, all conserved quantities will remain unaffected by the source of potential. Therefore, there can be no observables that can distinguish between ψ_0 and ψ_α. To check, we may compute the average of, say, the momentum p in the two states ψ_0 and ψ_α. This gives, as expected, $\langle p\rangle_0 = \langle p\rangle_\alpha$. For the average of the kinetic energy we find $\langle p^2/2m\rangle_0 = \langle p^2/2m\rangle_\alpha$, and a similar relation holds for the average of all other moments of momentum. Thus we see that the dynamical description of the potential effect, if possible at all, must involve new observables.

In order to bring the question of new dynamical observables into sharper focus we shall investigate an experiment which is a modification of the vector potential experiment described in Section 2. Consider a set-up consisting of a hollow diffraction grating g so constructed that a set of solenoids s can be placed inside, as shown in Fig. 1. The slit width is w, the slit spacing is l, and the total number of slits is N_g. All the solenoids carry the same current, and the experiment is repeated for different values of that current. Electrons so prepared that the y-component of their momentum is practically zero are incident on the slits from the left. In the following we shall speak only of the y-component of momentum.

When the current in the solenoids is zero, the resulting momentum distribution is well known; the observed values of momentum are integral multiples of the unit $p_0 = h/l$. When a current flows, the allowed momentum values are $(n + \alpha)p_0$, where α is Φ (modulo ch/e), Φ being the magnetic flux trapped in a single solenoid. To see this, observe that the flux introduces an extra relative phase α between the waves emanating from two successive slits and thus that the directions in which constructive interference occurs are shifted from

$$\theta_0 = \arcsin \frac{n\lambda}{l}$$

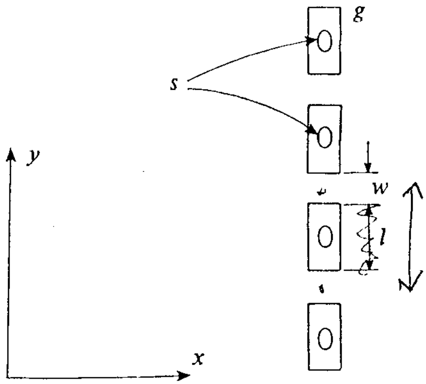
$p \sin\theta_0 = \frac{n\lambda}{l} p = \frac{nh}{l} = n p_0$

to

$$\theta_\alpha = \arcsin \frac{(n + \frac{\alpha}{2\pi})\lambda}{l}$$

$p \sin\theta_\alpha = (n + \frac{\alpha}{2\pi}) p_0$

The purpose of the experiment is to investigate the forceless interaction between the electron and the set of solenoids. That such an interaction exists



l = periodicity of potential

Figure 1

is shown by the alteration of the electron momentum when the solenoids are activated. In order to trace the momentum transfer as an electron passes through the slits, we must consider the momentum of the grating, p_g, and the momentum of the set of solenoids p_s. The solenoids are imagined to form a rigid system which is free to move with respect to the grating. We further assume that there is no interaction between the solenoids and the grating. Therefore, the grating can exchange momentum with the electron only. If the extension of the grating is sufficiently large compared to the region where the electron can be found (that is, the number of slits that the electron strikes, N_e, is small compared to the total number of slits, N_g), the interaction between the grating and the electron is periodic in the y-coordinate of the grating. Thus the grating can exchange momentum with the electron only in integral multiples of p_0. Conservation of momentum implies that the solenoids must absorb the residual momenta of the form $(m - \alpha)p_0$, where m is an integer.

Since the solenoids are confined to regions smaller than l, the uncertainty in the momentum p_s must be large compared to p_0. Thus, even for the most favorable case, i.e. $\alpha = \frac{1}{2}$, the exchanged momentum can be small compared to the initial uncertainty in p_s. However, if we repeatedly send electrons through the set-up, we expect to accumulate arbitrarily large changes in p_s. We shall suppose the solenoids to be so massive that they suffer negligible change in position during the time when the interaction with all the electrons takes place. Thus, the vector potential produced by the solenoids remains essentially unchanged. Since the electron-solenoid interaction is mediated by this vector potential, the interaction is the same for each electron. It would appear, therefore, that the change in the momentum p_s after N passages may be estimated by assuming that each individual passage causes an independent random exchange of at least plus or minus $\frac{1}{2}p_0$, i.e. $\Delta p_s \sim \sqrt{N}\frac{1}{2}p_0$. By choosing N large enough we can obtain an arbitrarily large Δp_s. There is no contradiction between the assumption that the solenoids suffer negligible change of position during the passage of the electrons and the observability of Δp_s, since Δp_s may be observed by allowing the solenoids to drift for a long time after all the electrons have passed.

On the other hand, we may argue on the basis of the correspondence principle that such a Δp_s should not occur. Since the set of solenoids exerts no force on the electrons, in classical theory it will cause no change of the momentum of the electrons. The solenoids do not lose momentum by radiation, since they are massive. Hence, the momentum of the solenoid array does not change.

Thus, quantum theory seems to predict that the electron-solenoid interaction produces changes in the momentum of the set of solenoids that are large compared to the quantum uncertainty h/l, while classical theory predicts no change at all.

In order to resolve the grating-solenoid paradox presented above we re-examine our notions concerning the change in the momentum distribution produced by collisions. In doing so we shall uncover some general features that characterize interactions in the quantum domain. It is convenient to introduce a characteristic set of functions of momentum $\mathscr{F}_n(p)$ whose average values determine the momentum distribution completely; the functions $\mathscr{F}_n(p)$ can usually be chosen so that they satisfy simple equations of motion during the entire collision. A familiar example is provided by a potential interaction in which case

$$\frac{d}{dt}\langle p\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle$$

similar equations hold for the higher moments of momentum $\langle p^n\rangle$.

We consider a special case of a forceless interaction: the gradient of the potential vanishes in a region S; the particle is so constrained that the probability of finding it outside S is zero. Then, in the classical case

$$\frac{d}{dt}\mathscr{F}_n(p) = \frac{d}{dp}\mathscr{F}_n(p)\cdot\frac{dp}{dt} = 0$$

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$$\frac{d}{dt}\mathscr{F}_n(p) = \frac{d}{dp}\mathscr{F}_n(p)\cdot\frac{dp}{dt} = 0$$

since in S

$$\frac{dp}{dt} = -\frac{\partial V}{dx} = 0$$

Thus, the momentum distribution for any ensemble of particles moving in force-free regions remains unchanged.

In quantum theory a more limited result holds; as shown in Appendix A the time derivative of each of the moments $d\langle p^n \rangle/dt$ vanishes.

If the set of moments exhausted a characteristic set, the time derivative of the momentum distribution function would be zero in quantum theory as well as in classical theory. However, as the grating-solenoid paradox indicates, in quantum theory the momentum distribution function may change in cases of forceless interaction. We conclude that the set of moments does not exhaust the characteristic set. In the following we shall explore the properties of the additional members of the characteristic set. In particular, we shall investigate the qualitative difference in the quantum and classical equations of motion for the additional variables. That there is such a difference is evident since any additional variables are constant in the classical case, as shown above, whereas in the quantum case such variables may change.

To find the missing members of the characteristic set, we return to the grating-solenoid paradox. To begin with, we observe that the moments argument of the preceding paragraph may be applied to the momentum distribution of the solenoid array, and therefore $\langle p_s^2 \rangle$ remains unchanged (see Appendix B). Hence, we must have been misled by the random walk argument which implied that $\langle p_s^2 \rangle$ increases indefinitely.

How could the random walk argument fail? We have assumed that as each electron goes by it exchanges momentum with the solenoid array and that the succeeding exchanges are independent of each other. The assumption of independence implies that the momentum exchange is governed by either a random or a linear walk. In a linear walk Δp_s would have been proportional to the number of collisions rather than to its square root; obviously, this would only sharpen the paradox. We may then feel forced to abandon the assumption of independence, which would imply that the solenoid array has some sort of quantum memory. The conventional way of understanding a concept of this type would be to study in detail the evolution of the wave function for the whole system.

There is, however, the possibility of retaining the independence assumption if we abandon the classical notion that every interaction must be analyzed in terms of exchange of momentum. In other words, there may exist conserved quantities that are exchanged between the electrons and the solenoid array, and for which succeeding exchanges are independent of each other. We may further hope that these variables will complete the characteristic set $\{\mathscr{F}(p)\}$. In fact, such variables do exist and they provide the most natural description of forceless interactions.

How are we to find these variables? Our basic clue is that in terms of them the random walk argument should be free of paradox. We therefore require such a variable to fulfill the following condition: its change after many collisions should be comparable to its change after one collision, so that the associated changes in momentum could remain bounded. Clearly, this condition suggests a bounded function of momentum.

Consider the variable p modulo p_0, that is

$$p = np_0 + p(\mathrm{mod}\, p_0)$$

where n is an integer and the remainder $p(\mathrm{mod}\, p_0)$ satisfies

$$0 \leqslant p(\mathrm{mod}\, p_0) < p_0$$

Note that a random or linear walk of ordinary momentum can be thought of as taking place on an infinite straight line, whereas a walk of modular momentum can be thought of as taking place on a circle.

To see that the modular momentum helps to characterize the momentum distributions before and after the forceless interactions in the grating-solenoid experiment, consider what happens to the modular momentum of an electron as it passes through the set-up. Since the initial value of the momentum itself is zero, the modular momentum has a definite value, zero, before the electron reaches the grating. Suppose first that no current flows in the solenoids. Then the value of the momentum after the electron has passed the grating can be $n(h/l)$ only; thus $p(\mathrm{mod}\, h/l)$ is left unchanged. With a current flowing, the electron emerges with one of the momentum values $p = (n + \alpha)h/l$, where $\alpha = \Phi(\mathrm{mod}\, ch/e)$. Hence, the electron modular momentum $p(\mathrm{mod}\, h/l)$ has been shifted to the value $\alpha(h/l)$. The modular momentum of the solenoid array has been shifted correspondingly. In Appendix C we investigate this shift and show that the set of all modular momenta is characteristic. In contrast to the moments, the modular momentum is affected by the forceless interaction in such a way that its average value is changed.

When the electron interacts with the grating alone (that is, when the solenoid current is turned off), the moments of both electron and grating momentum distribution are changed, while $p(\mathrm{mod}\, h/l)$ is unchanged. Consequently, the random walk argument applied to the grating correctly predicts that the grating will eventually display macroscopic motion provided the number of electrons is sufficiently large. The interaction of the electron with the solenoids has the opposite property: it changes the modular momenta of both electron and solenoid array while it does not change the moments of momentum.

4. *General Properties of Modular Variables*

We shall now discuss some of the fundamental properties of modular variables. We first investigate the equation of motion of modular momentum under the influence of an external force. For simplicity we discuss a one-dimensional problem with the Hamiltonian $H = p^2/2m + V(x)$.

Consider the operator related to $p(\text{mod}\, p_0)$

$$A = \frac{p_0}{2\pi}\sin 2\pi \frac{p}{p_0}$$

From the definition of A we see that in the limit $p_0 \to \infty$, A approaches p. Thus, the equation of motion for A will approach the Newtonian equation of motion for $p_0 \to \infty$.

In classical theory,

$$\frac{d}{dt}A = \dot{p}\cos 2\pi \frac{p}{p_0} = -\frac{dV}{dx}\cos 2\pi \frac{p}{p_0} \tag{4.1}$$

Thus, when the force $F = -(dV/dx)$ is equal to zero at the position of the particle, A is a constant of the motion for all p_0.

In quantum theory, the situation is strikingly different. To see this, calculate $d/dt\langle A\rangle$. One gets

$$\frac{d}{dt}\left\langle \frac{p_0}{2\pi}\sin 2\pi \frac{p}{p_0}\right\rangle$$

$$= \frac{p_0}{2\pi}\frac{d}{dt}\int \psi^*(x,t)\frac{\psi(x+l,t)-\psi(x-l,t)}{2i}dx \qquad \left(l = \frac{h}{p_0}\right)$$

Using the Schrödinger equation this gives

$$\frac{d}{dt}\langle A\rangle = -\int \frac{\psi^*(x)\psi(x+l)+\psi^*(x+l)\psi(x)}{2}\frac{V(x+l)-V(x)}{l}dx$$

For wave functions with sharp maxima at x_0 and $x_0 + l$ we may remove the potential from the integral, evaluating it at the maxima. This gives

$$\frac{d}{dt}\langle A\rangle = -\frac{V(x_0+l)-V(x_0)}{l}\left\langle \cos 2\pi \frac{p}{p_0}\right\rangle \tag{4.2}$$

Note that in the limit $l \to 0$ the equation of motion for $\langle A\rangle$ becomes

$$\frac{d}{dt}\langle p\rangle = \int \psi^*(x)\left(-\frac{dV}{dx}\right)\psi(x)\,dx = \left\langle -\frac{dV}{dx}\right\rangle = \langle F(x)\rangle$$

because

$$\frac{p_0}{2\pi}\sin 2\pi\frac{p}{p_0} \to p, \qquad \frac{V(x+l)-V(x)}{l} \to \frac{dV}{dx}$$

$$\frac{\psi^*(x)\psi(x+l)+\psi^*(x+l)\psi(x)}{2} \to \psi^*(x)\psi(x)$$

For finite l,

$$\frac{V(x_0+l)-V(x_0)}{l}$$

appears as the quantity that determines the effect of the external field. Thus, the negative of this difference quotient of the potential function may be considered as a generalized non-local force which is responsible for changing the generalized non-local momentum A. In the absence of a potential difference between the points x_0 and $x_0 + l$, the dynamical variable A is conserved. Otherwise, there may be an 'exchange' of A between the object and the source of potential.

The quantum equation of motion for the average of ordinary momentum is identical to the classical equation of motion for p, but as we see by comparing equations (4.1) and (4.2) the classical and the quantum equations of motion for modular momentum are essentially different. Thus, we cannot draw conclusions from the correspondence principle about averages of modular momentum. In particular, the classical variable A may be a constant of the motion under conditions when the quantum variable A is not conserved. This happens when the interaction between the particle and the source of potential is purely non-local.

It is easy to show that the expectation values of the variable A in the set of states ψ_α where

$$\psi_\alpha(x) = f(x) + \exp(i\alpha) f(x - l)$$

is

$$\langle A \rangle_\alpha = \frac{\hbar}{2l} \sin \alpha$$

Consequently, the modular momentum is sensitive to the relative phase α between the two wave packets contained in ψ_α.

We thus see that even though the potential effect is a typical quantum effect with no classical analog, it can be described in a physical language which is very similar to that used in describing phenomena in the classical domain. Indeed, the potential effect now appears as a straightforward example of exchange of a non-local dynamical quantity, the modular momentum, with no exchange of local quantities. The possibility of such physical situations arises from the fact that the generalized non-local force may be non-vanishing even though the local force is zero wherever the particle can be found.

The difference in the structure of the equations of motions is one significant expression of the fact that modular momentum plays qualitatively different roles in classical and quantum theory. Another way to appreciate this distinction is to compare Poisson brackets and commutators for modular quantities.

Consider the observables

$$A = \frac{p_0}{2\pi} \sin 2\pi \frac{p}{p_0}, \qquad B = \frac{x_0}{2\pi} \sin 2\pi \frac{x}{x_0}$$

We see that the Poisson bracket of A and B is

$$\{A, B\} = -\cos\frac{p}{p_0}\cos\frac{x}{x_0}$$

and that the commutator of A and B is

$$[A, B] = 0$$

if $x_0 p_0 = h$. Thus, the variables A and B may commute even though the Poisson bracket of the corresponding classical variables is different from zero.

This result shows that the connection between Poisson brackets and commutators breaks down for modular varaiables. In fact, while the equation

$$\{F(x), G(p)\} = 0$$

has no non-trivial solutions, the equation

$$[F(x), G(p)] = 0 \tag{4.3}$$

has the following set of non-trivial solutions:

$$F(x) = F(x+nl), \qquad G(p) = G\left(p + n\frac{h}{l}\right) \qquad (n = 1, 2, \ldots)$$

To see this we write

$$G(p) = \sum_n g_n \exp\,[in(pl/\hbar)]$$

and we have

$$[F(x), G(p)] = [F(x), \sum_n g_n \exp\{in(pl/\hbar)\}$$

but

$$[F(x), \exp\{in(pl/\hbar)\}] = F(x)\exp\,[in(pl/\hbar)] - \exp\,[in(pl/\hbar)]\,F(x)$$
$$= \exp\,[in(pl/\hbar)]\{\exp\,[-in(pl/\hbar)]\,F(x)\exp\,[in(pl/\hbar)] - F(x)\}$$

This is equal to zero since

$$\exp\,[-in(pl/\hbar)]\,F(x)\exp\,[in(pl/\hbar)] = F(x-nl) = F(x)$$

It is easily shown that these periodic functions are the only solutions of equation (4.3).

The commutativity of the variables $\bar{x} = x(\text{mod}\, l)$ and $\bar{p} = p(\text{mod}\, h/l)$ suggests the introduction of a representation in which the operators $\bar{x}$ and $\bar{p}$ both have a diagonal form. This modular representation will be investigated elsewhere†. Introducing the operators M and N which take on only integer eigenvalues, we may write

$$p = M\frac{h}{l} + \bar{p}, \qquad x = Nl + \bar{x}$$

† In an article 'Further Developments of the Modular Variables' (to be submitted for publication in the *Physical Review*, 1969).

Since

$$[M, \exp\{2\pi i(\bar{x}/l)\}] = i \exp[2\pi i(\bar{x}/l)],$$
$$[N, \exp\{i(\bar{p}l/\hbar)\}] = -i \exp[i(\bar{p}l/\hbar)]$$

we have

$$\Delta\bar{x}\Delta M \sim l \qquad \Delta\bar{p}\Delta N \sim \frac{h}{l}$$

We thus see that the modular variables suggest a generalization of the familiar uncertainty relations $\Delta x \Delta p \sim h$. The need for such a generalization for cases involving non-overlapping wave packets is evident, since $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ is independent of the number of wave packets and the relative phase between them.

5. *Modular Energy*

In previous sections we have investigated the significance of modular momentum. Obviously, there are situations in which other conserved modular variables are exchanged. We now discuss an example of non-local forceless exchange of modular energy.

Consider a particle at a fixed position. The particle has spin $\frac{1}{2}$ and magnetic moment μ. It is under influence of a static magnetic field in the z-direction, $B_z{}^0$. In addition, the particle is perturbed by a weak time-dependent magnetic field in the x-direction, $B_x(t)$ (see Fig. 2). The basic frequency of $B_x(t)$, $\nu = 1/T$ is chosen to be $2\bar{\nu}$, where $\bar{\nu}$ is the transition frequency of the spin, $B_z\mu/2\hbar$. Since the Fourier transform of B_x includes only multiples of ν, none of which equals $\bar{\nu}$, no real transition will take place, and the energy of the system consisting of the static magnet plus the spin will be the same after B_x is switched off as it was before B_x was switched on (we have assumed that the period T' during which B_x is different from zero, is very large compared to T). In the following discussion we consider the experiment to be a 'collision' between system I (static magnet + spin) and system II [array of magnets producing $B_x(t)$]. In this language we can say that no exchange of energy between systems I and II took place in the collision.

Consider now a modified experiment in which a third system III consisting of an array of magnets producing a time dependent magnetic field $B_z(t)$ (see Fig. 3) is added.

As indicated in Fig. 3, B_z and B_x are arranged so that they do not overlap at any time. Nevertheless, it is obviously possible to choose B_z in such a way that the spin of the particle will have flipped after T', i.e. after both collisions have taken place. This result, which is easily checked, raises some interesting questions concerning the over-all conservation of energy.

In the collisions, the energy of system I has been changed by the definite amount $h\bar{\nu}$. This energy must have come from system II or system III, or both. But system II, being periodic in time, could have transferred to system

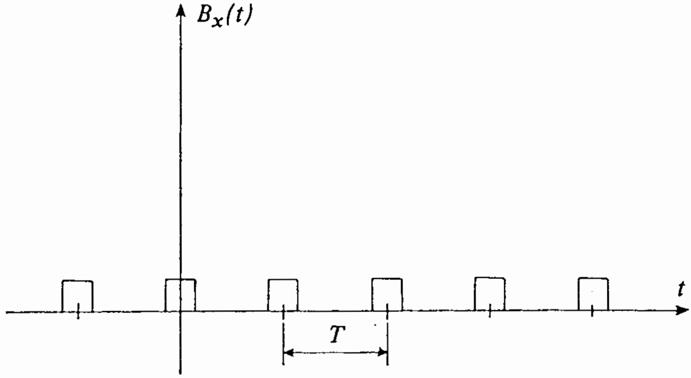

Figure 2

I only an integer multiple of $h\nu$. Any such multiple differs from $h\bar{\nu}$ by at least $\frac{1}{2}h\nu = h\bar{\nu}$. Thus, system III must have contributed to the energy of system I at least $\pm h\bar{\nu}$. Just as in the example discussed in Section 3, this result is rather paradoxical.

Magnet III does not change at all the z-component of the particle's spin. Thus in each of its collisions with system I, magnet III leaves unchanged the Hamiltonian of system I (which is proportional to σ_z). Therefore, it appears that system III could not affect the energy of system I in any way. Indeed, in any analogous classical situation, this argument would be sufficient to prove that no exchange of energy takes place between systems I and III. Yet, as we have seen, at least an energy $h\bar{\nu}$ has to be exchanged.

On the background of the previous discussion, the solution of the paradox should be transparent. The classical argument that no energy can be exchanged between systems III and I is applicable only to the moments of the energy distribution of both systems (which indeed remain unaffected

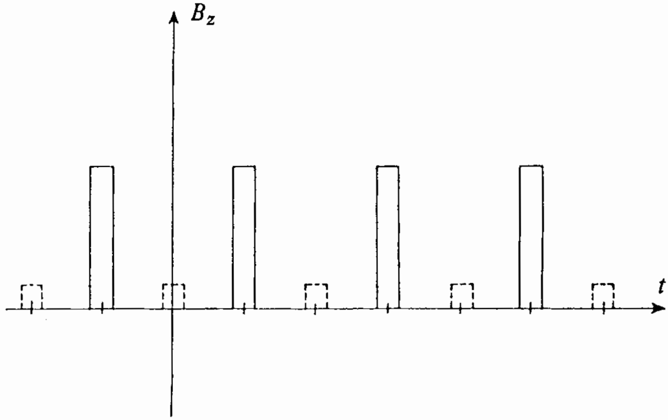

Figure 3

in the collisions). It is not applicable, however, to the systems' modular energy whose exchange is determined in a non-local way by the interaction between systems I and II. Note that here the non-locality is in time rather in space as was the case in the grating-solenoid experiment. In other words, if system II does not interact with system I, exchange of modular energy does not take place, while if system II does interact with system I, the exchange does take place, although the two interactions happen at different times.

Imagine now a similar experiment in which system I consists of a large number of spins. As we have seen, the moments of the energy distribution of system III remain conserved in all collisions. Thus we may think of system III as a 'catalyzer' causing large changes in the behavior of systems I and II without accumulating a large effect on its own energy distribution.

6. *Conclusion*

We have shown that the modular variables are suitable for the description of quantum features of interaction. In particular, we have demonstrated the importance of their non-local characteristics in the quantum domain. In forthcoming articles we will show how these variables lead to a new view of well-known but not fully understood basic quantum effects, such as interference. We also show how these variables suggest new interesting quantum effects.

Appendix A

We have that

$$i\hbar \frac{d}{dt} p^n = [p^n, H] = [p^n, V] = p[p^{n-1}, V] + [p, V]p^{n-1}$$

Then

$$i\hbar \frac{d}{dt} \langle p^n \rangle = (p\psi, [p^{n-1}, V]\psi) + (\psi, [p, V]p^{n-1}\psi)$$

The conditions of our problem imply that the wave function ψ vanishes outside the region S. The $p^m\psi$ must also vanish outside S. We now argue by induction that any expression of the form $(\phi, [p^k, V]\chi)$ vanishes for all ϕ and χ vanishing outside S, for $k \geqslant 1$. This follows immediately from the identity

$$\begin{aligned}(\phi, [p^k, V]\chi) &= (p\phi, [p^{k-1}, V]\chi) + (\phi, [p, V]p^{k-1}\chi) \\ &= (\phi', [p^{k-1}, V]\chi) + (\phi, [p, V]\chi')\end{aligned}$$

and the observation that

$$i(\phi, [p, V]\chi') = \hbar\left(\phi, \frac{\partial V}{\partial x}\chi'\right),$$

which vanishes since the gradient of V is zero where ϕ and χ' are non-zero.

Appendix B

Conservation of total momentum implies that $\langle (p_g + p_s + p_e)^2 \rangle$ is unchanged. Since the interaction between the grating electron system and the solenoids is forceless, $\langle (p_s + p_e)^2 \rangle$ is unchanged. But

$$\langle (p_g + p_s + p_e)^2 \rangle = \langle p_s^2 \rangle + \langle (p_g + p_e)^2 \rangle + 2\langle p_s(p_g + p_e) \rangle \qquad \text{(A.1)}$$

In order to use equation (A.1), we shall show that the cross-term

$$\langle p_s . (p_g + p_e) \rangle$$

remains zero throughout. We assume that initially $\langle p_e \rangle = \langle p_g \rangle = 0$ and the system is uncorrelated, which implies $\langle p_g . p_s \rangle = 0$ and $\langle p_e . p_s \rangle = 0$. Since both the grating and the solenoids are very massive, we can describe their effects on the electrons to a good approximation in terms of potential functions depending only on the coordinates of the electrons. As a result, the final wave function for the whole system still factors:

$$\Psi = \prod_n \psi^s \psi^g \psi_n^e$$

Therefore, $\langle p_s . (p_g + p_e) \rangle = \langle p_s \rangle . \langle p_g + p_e \rangle$ even after collision. But $\langle p_g + p_e \rangle$ does not change because the interaction is forceless, so the cross-term remains zero. Hence, according to equation (A.1), $\langle p_s^2 \rangle$ remains unchanged.

Appendix C

Let $\bar{p} = p(\text{mod}\, h/l)$. Then it is easy to check that

$$\overline{p_2} = \overline{\overline{p_1 + p_2} - \overline{p_1}}$$

where p_1 and p_2 are the momenta of the two interacting parts of a closed system. Since $p = p_1 + p_2$ and therefore $\bar{p} = \overline{p_1 + p_2}$ remain unchanged as a result of the interaction, we get

$$\overline{p_2{}^{\text{final}}} = \overline{\bar{p} - \overline{p_1{}^{\text{final}}}}$$

Thus, if initially we know $\overline{p_1}$ and $\overline{p_2}$ and then after the interaction is over we measure $\overline{p_1{}^{\text{final}}}$, the above equation permits us to predict $\overline{p_2{}^{\text{final}}}$. Applying this result to the electron-solenoid interaction we see that the change in $\bar{p}_{\text{solenoid}}$ is determined.

It is obvious that $\bar{p}$ determines $\exp[i(pl/\hbar)]$ and $\exp[-i(pl/\hbar)]$, and vice versa. Since the average of $\exp[i(pl/\hbar)]$ is the Fourier transform of the momentum distribution function, it follows that these averages for all l specify completely any distribution which can be Fourier transformed.

References

Aharonov, Y. and Bohm, D. (1959). *Physical Review*, **115**, 485.

Boersch, H., Hamisch, H., Grohmann, D. and Wohllenben, D. (1960). *Zeitschrift für Physik*, **159**, 397.

Boersch, H., Hamisch, H., Grohmann, D. and Wohllenben, D. (1961a). *Zeitschrift für Physik*, **164**, 55.

Boersch, H., Hamisch, H., Grohmann, D. and Wohllenben, D. (1961b). *Zeitschrift für Physik*, **165**, 69

Boersch, H., Hamisch, H., Grohmann, D. and Wohllenben, D. (1962a). *Zeitschrift für Physik*, **167**, 72.

Boersch, H., Hamish, H., Grohmann, D. and Wohllenben, D. (1962b). *Zeitschrift für Physik*, **169**, 263.

Bohr, N. (1949). Discussion with Einstein on Epistemological Problems in Atomic Physics. In: *Albert Einstein—Philosopher-Scientist*, p. 199, ed. Schilpp, P. A. The Library of Living Philosophers, Inc., New York.

Chambers, R. G. (1960). *Physical Review Letters*, **5**, 3.

Fowler, H. A., Marton, L., Simpson, Y. A. and Suddeth, J. A. (1961). *Journal of Applied Physics*, **32**, 1153.

Furry, W. H. and Ramsey, N. F. (1960). *Physical Review*, **118**, 623.

Jaklevic, R. C., Lambe, J., Silver, A. H. and Mercereau, J. E. (1964a). *Physical Review Letters*, **12**, 159.

Jaklevic, R. C., Lambe, J., Silver, A. H. and Mercereau, J. E. (1964b). *Physical Review Letters*, **12**, 274.

Marton, L., Arol Simpson, J. and Suddeth, J. A. (1954). *Review of Scientific Instruments*, **25**, 1099.

Möllenstedt, G. and Bayh, W. (1962a). *Die Naturwissenschaften*, **49**, 81.

Möllenstedt, G. and Bayh, W. (1962b). *Physikalische Blätter*, **18**, 299.

Werner, F. G. and Brill, D. R. (1960). *Physical Review Letters*, **4**, 344.