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Deterministic Quantum Interference Experiments†

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Abstract

We explain why quantum interference may pertain to a single degree of freedom and demonstrate how it can be exhibited by deterministic experiments involving a single particle.

1. *Introduction*

From the beginning of quantum theory the problem of interference has been a central issue. It was clear that interference in the quantum domain differs in fundamental respects from classical interference. In classical physics, interference is exhibited only by systems with many or an infinite number of degrees of freedom, e.g. sound waves or electromagnetic waves. In quantum physics, a system with a single degree of freedom may exhibit interference. This difference has presented the interpretation of the quantum type of interference with great difficulties.

Consider, for example, the familiar two-slit interference experiment. For a classical wave, the description of the phenomenon is straightforward. In

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particular, there is no difficulty in understanding that the resulting pattern depends on contributions from both slits, since parts of the wave, corresponding to different degrees of freedom, pass through each slit. In the case of an electron or a photon, however, only a single degree of freedom is involved, and it is difficult to understand how there can be contributions from both slits.

Furthermore, although quantum interference is exhibited by a single degree of freedom, both conceptual and actual quantum interference experiments always have involved a large number of particles. In the two-slit experiment the interference pattern emerges only when many particles have been registered on the photographic plate. Even though the interference property pertains to the individual particle, no information regarding this property can be deduced from a single spot on the plate resulting from just one particle passing the slits. Thus, it has become an accepted view that quantum interference phenomena are inseparable from the statistical aspects of quantum theory.

In this paper, we want to explain why quantum interference may pertain to a single degree of freedom, and to demonstrate how it can be exhibited by experiments involving a single particle.

2. *Deterministic Experiments*

Our approach is based on the notion of deterministic experiments. We call an experiment a deterministic experiment when we measure only physical variables for which the state of the system under investigation is an eigenstate. For any state ψ it is possible to ask the following question: What is the set of hermitian operators A for which ψ is an eigenstate, i.e. for which $A(t)\psi(t)=\lambda(t)\psi(t)$? We call such operators eigenoperators of ψ. A measurement of an eigenoperator A would not lead to a collapse of the wave function, since the wave function was to begin with an eigenstate of the operator that was measured. We can perform a succession of such experiments in time. No statistics is involved; the results are completely predictable, i.e. the experiments are deterministic.

As an example, consider a spin-$\frac{1}{2}$ particle placed in a magnetic field pointing in the x-direction and with $\sigma_z=+1$ initially. If some time t later we measure σ_z again, the result is not predictable; we get fluctuations and the experiment is therefore not a deterministic one. However, if instead we rotate our apparatus and measure the spin in the yz-plane in a direction $\alpha=\omega t$, where ω is the Larmor frequency, then the result is predictable and we have a deterministic experiment.

As a second example, consider a free particle of mass m which at $t=0$ is in the state $\psi=C\exp[-x^2/(\Delta x)^2]$. This state is an eigenstate of the hermitian operator $A(0)=x^2+p^2$. At time t we can measure the eigenoperator $A(t)=(x-pt/m)^2+p^2$ with a predictable outcome, and thus we have again a deterministic experiment.

3. *Eigenoperators for the Two-Slit Experiment*

Let $\psi(x,y,t) = a_1\psi_1(x,y,t) + a_2\psi_2(x,y,t)$ be the wave function of a particle emerging from the two slits, where $\psi_2(x,y,t) = \psi_1(x-l,y,t)$ and l is the distance between the two slits. For simplicity we choose the y-dependence of ψ so that immediately after the particle emerges from the slits the overlap of ψ_1 and ψ_2 is negligible. Our purpose is to find eigenoperators $A(a_1, a_2)$ which satisfy the condition

$$A(a_1,a_2)\,[a_1\psi_1(x) + a_2\psi_1(x-l)] = \lambda[a_1\psi_1(x) + a_2\psi_1(x-l)]$$

for all $\psi_1(x)$ which vanish for $|x| \geqslant l/2$.

Since for any choice of ψ_1 the action of A is analogous to that of a spin operator, we expect the A operators to form a spin algebra. In other words, there should exist three operators $\bar{\sigma}_1$, $\bar{\sigma}_2$, $\bar{\sigma}_3$ which satisfy the relations

$$[\bar{\sigma}_i, \bar{\sigma}_j]_- = i\epsilon_{ijk}\bar{\sigma}_k, \qquad [\bar{\sigma}_i, \bar{\sigma}_j]_+ = 2\delta_{ij}$$

As is easily verified, these conditions are satisfied by

$$\bar{\sigma}_3 = \frac{\sin(\pi x/l)}{|\sin(\pi x/l)|}$$

$$\bar{\sigma}_1 = \cos\frac{pl}{\hbar} - \sin\frac{pl}{\hbar}\frac{\sin(\pi x/l)}{|\sin(\pi x/l)|}$$

$$\bar{\sigma}_2 = \sin\frac{pl}{\hbar} + i\cos\frac{pl}{\hbar}\frac{\sin(\pi x/l)}{|\sin(\pi x/l)|}$$

It is obvious that measurement of $\bar{\sigma}_3$ tells which slit the particle went through, and that the other two operators give information about the relative phase between the two wave packets. For example, if the particle goes through the first slit, then $\bar{\sigma}_3 = 1$, while if the particle goes through both slits with equal probabilities and with relative phase α, i.e.

$$\psi = \frac{1}{\sqrt{2}}\{\exp[i(\alpha/2)]\psi_1 + \exp[-i(\alpha/2)]\psi_2\},$$

then $\bar{\sigma}_1\cos\alpha + \bar{\sigma}_2\sin\alpha = 1$. For any other choice of a_1 and a_2 we can make, just as in the analogous spin problem, a corresponding choice in the space of the $\bar{\sigma}$-operators, and thus define the appropriate eigenoperator.

The eigenoperators of the states produced in the two-slit experiment are functions of the modular momentum and position introduced in an earlier paper (Aharonov *et al.*, 1969). As was shown in that paper, the equation of motion for modular momentum is non-local. This fact, which implies that the equations of motion for $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are also non-local, explains the non-locality involved in quantum interference experiments. In such experiments, the motion of a single degree of freedom is affected by both slits, even though the potential describing the interaction between the particle and the slits is local, i.e. a function of x only. This is in complete accordance with the fact that the non-locality in the time evolution of modular momentum is present even in the case of a local Hamiltonian.

Thus we see that the difference between classical and quantum interference has a dynamical origin. Classical dynamics, which is generated by the Poisson brackets, is always local. Therefore, the only way that two slits with local potential can affect a classical system passing through them is by each interacting with the degrees of freedom in its immediate vicinity. Quantum dynamics, on the other hand, is generated by the commutators which lead to a non-local behavior of interference eigenoperators, and hence a single degree of freedom can exhibit interference.

4. *Deterministic Interference Experiments*

The analogy between the $\bar{\sigma}$-operators and the ordinary spin operators suggests that deterministic interference experiments should be analogous to the Stern-Gerlach experiment. For example, we should be able to distinguish between the orthogonal states $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$ in an experiment performed on a single particle by producing a deflection proportional to the value of $\bar{\sigma}_1$. Customary experiments do not distinguish between these two states by observations made on a single particle. The desired deflection for our deterministic experiment may be produced by the Hamiltonian

$$H = H_0 - g(t)\,\bar{\sigma}_1\, z$$

where $g(t)$ is switched on immediately after the particle has passed through the slits. This interaction will cause an acceleration in the positive z-direction if the state is $\psi_1 + \psi_2$ and an acceleration in the negative z-direction if the state is $\psi_1 - \psi_2$. If $g(t)$ is sufficiently strong, the resulting z-deflections will separate the two states completely, just as the analogous interaction does in the Stern-Gerlach experiment for the ordinary spin.

The above Hamiltonian is consistent with the principles of non-relativistic quantum mechanics. Nevertheless, since the type of non-local interaction contained in the Hamiltonian violates the causality condition imposed by relativity theory, it is interesting to investigate whether one can perform deterministic experiments which involve only local interactions. That this is indeed the case is shown by the following two examples.

Example I. We use a procedure suggested by Lamb (1969). The crux of this procedure is to find a potential for which one of the two states to be separated is an eigenstate while the other is not. If this potential is switched on suddenly the first state will be 'trapped', while the second state, after a sufficient time, will have drifted away. To find such a potential is straightforward. We simply solve the Schrödinger equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{x}) + V(\mathbf{x})\,\psi(\mathbf{x}) = E\psi(\mathbf{x})$$

for the potential and get

$$V(\mathbf{x}) = \frac{[E + (\hbar^2/2m)\nabla^2]\psi}{\psi}$$

It is easily seen that a direct application of this procedure is not useful for the two interference states under consideration, i.e. $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$. Indeed, as long as ψ_1 and ψ_2 do not overlap, any $V(\mathbf{x})$ which traps $\psi_1 + \psi_2$ will also trap $\psi_1 - \psi_2$. Let $H(\psi_1 + \psi_2) = E(\psi_1 + \psi_2)$. Then it follows that $H\psi_1 = E\psi_1$, and thus that $H(\psi_1 - \psi_2) = E(\psi_1 - \psi_2)$. To see this, write

$$H\psi_1 = E\psi_1 + \delta\psi_1 \qquad \text{and} \qquad H\psi_2 = E\psi_2 + \delta\psi_2$$

Thus

$$H(\psi_1 + \psi_2) = E(\psi_1 + \psi_2) + \delta\psi_1 + \delta\psi_2$$

and hence

$$\delta\psi_1 + \delta\psi_2 = 0$$

But since H is a local operator and ψ_1 and ψ_2 do not overlap, $\delta\psi_1$ and $\delta\psi_2$ do not overlap either, and therefore the last equation implies that $\delta\psi_1 = \delta\psi_2 = 0$.

This difficulty is overcome by bringing the two wave packets together before switching on the potential. This can be done by applying opposite forces to the two packets. Consider for simplicity a one-dimensional case where initially

$$\psi_1 = A\exp\left(-\frac{(x-a)^2}{(\Delta x)^2}\right) \qquad \text{and} \qquad \psi_2 = A\exp\left(-\frac{(x+a)^2}{(\Delta x)^2}\right)$$

The separation $l = 2a$ is assumed to be large compared to the width Δx. The opposite forces produce a relative momentum 2_0 between the two packets, which then become

$$\psi_1 = A\exp\left(-\frac{(x-a)^2}{(\Delta x)^2}\right)\exp(-ik_0 x) \text{and}\ \psi_2 - A\exp\left(-\frac{(x+a)^2}{(\Delta s)^2}\right)\exp(ik_0 x)$$

where $hk_0 = p_0$. After a period $T = am/hk_0$ the two packets overlap completely and the states $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$ are

$$\psi_1 + \psi_2 = A\exp\left(-\frac{x^2}{(\Delta x)^2}\right)\cos k_0 x$$

$$i(\psi_1 - \psi_2) = A\exp\left(-\frac{x^2}{(\Delta x)^2}\right)\sin k_0 x$$

(It is assumed that $k_0 \gg 1/\Delta x$ so that the dispersion of the packets during the period T can be neglected.)

With this preparation, Lamb's method may now be applied.

Example II. Consider the same one-dimensional situation as in the first example at the time when the two wave packets overlap. If the spread of the wave packets, Δx, is sufficiently large, $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$ may be regarded as eigenstates of the free Hamiltonian $H_0 = p^2/2m$. It is then possible to switch on adiabatically a potential $V(x) = \cos 2k_0 x$ which will remove the degeneracy of the two states $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$ and place $\psi_1 - \psi_2$ at the top of the first energy band of the periodic potential and $\psi_1 + \psi_2$ at the bottom of the second band (Kittel, 1953). If we then add a constant force $F(x) = F_0$, then $\psi_1 + \psi_2$ will be driven in the direction of the force, while $\psi_1 - \psi_2$ will be driven in the direction opposite to the force. In this way, a Stern-Gerlach type of measurement of eigenoperators for the two-slit interference experiment has been achieved.

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