Generalized Destruction Operators and a New Method in Group Theory.

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An important property [1] of the coherent states of harmonic oscillators is that they retain their character as eigenstates of the destruction operator under the influence of appropriate (in this case, linear c-number) driving forces. This means that there is a class of Hamiltonians which generate transformations of coherent states into themselves. Analysis of oscillator phase operators [2] has led to a class of states similar to the coherent states in that they are a nondegenerate, overcomplete set of eigenstates of an annihilation-type operator $U = EF(N)$, where $F(N)$ is a function of the number operator and E is the unit shift operator on the number basis, *i.e.* $E|n\rangle = (1-\delta_{0n})|n-1\rangle$.

In seeking Hamiltonians which generate transformations of these states into themselves, we were led to the following mathematical characterization of the problem.

Consider an operator A (not necessarily Hermitian) having a nondegenerate set of eigenstates $|\alpha\rangle$ spanning a given Hilbert space. It is easily shown that a necessary and sufficient condition for a Hamiltonian H to generate transformations of the set $|\alpha\rangle$ into itself is that

$$[A, H] = f(A)\,, \tag{1}$$

where $f(A)$ is some well-defined function of A. If H_1 and H_2 are two Hermitian operators satisfying eq. (1) with an $f_1(A)$ and $f_2(A)$, respectively, it is clear that $(\lambda_1 H_1 + \lambda_2 H_2)$ and $[H_1, H_2]$ satisfy

$$[A, \lambda_1 H_1 + \lambda_2 H_2] = \lambda_1 f_1(A) + \lambda_2 f_2(A)\,, \tag{2a}$$

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(1) See, for example, the article by R. J. GLAUBER: in *Quantum Optics and Electronics, Les Houches, 1964* (New York, 1965).

(2) E. C. LERNER, H. W. HUANG and G. E. WALTERS: *Journ. Math. Phys.*, **11**, 1679 (1970).

and

$$[A, [H_1, H_2]] = f_2(A)\frac{\mathrm{d}f_1(A)}{\mathrm{d}A} - f_1(A)\frac{\mathrm{d}f_2(A)}{\mathrm{d}A}\,. \tag{2b}$$

Thus the set of all H satisfying eq. (1) provides a representation of a Lie algebra.

Taking A of eq. (1) to be any one of the U-operators associated with oscillator phase, we find that the U-operators separate into two classes. One of these leads to Hamiltonians belonging to uninteresting Abelian Lie algebras. The second class belongs in turn (to within a multiplicative factor) to a one-parameter family of operators $A(k)$ defined by

$$A(k) = E\sqrt{\frac{N(k+1)}{N+k}}\,, \tag{3}$$

where k is a real number $k > -1$. The eigenstates of $A(k)$ with eigenvalue α are

$$|\alpha; k\rangle = \left[1 - \frac{|\alpha|^2}{k+1}\right]^{(k+1)/2} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+k+1)}{n!\,\Gamma(k+1)}} \left[\frac{\alpha}{\sqrt{k+1}}\right]^n |n\rangle\,, \tag{4}$$

where $|\alpha| < \sqrt{|k+1|}$. These are indeed overcomplete ([3]) and nondegenerate. The Hamiltonians associated with $A(k)$ are linear combinations of H_1, H_2, H_3, where

$$H_1 + iH_2 \equiv H_+ = H_-^+ = \sqrt{N(N+k)}\,E^+\,, \qquad H_3 = N + \left(\frac{k+1}{2}\right). \tag{5}$$

These satisfy the commutation relations of $O_{2,1}$ and the relation

$$H_3^2 - H_1^2 - H_2^2 = \tfrac{1}{4}(k^2 - 1)$$

For this case eq. (1) takes the form

$$[A, H_3] = A\,, \qquad [A, H_+] = \sqrt{k+1}\,, \qquad [A, H_-] = \frac{A^2}{\sqrt{k+1}}\,. \tag{7}$$

The objective behind the introduction of eq. (1) is thus achieved. It is now a simple matter to exhibit the dynamics associated with these Hamiltonians in terms of the eigenstates of $A(k)$. As in the familiar case of the coherent states and their associated Hamiltonians, eigenstates of $A(k)$ are transformed into eigenstates of $A(k)$ with the transformed eigenvalues determined by eq. (7). In fact, in the limit $k \to \infty$, $A(k)$ becomes the usual destruction operator and $|\alpha; k\rangle$ becomes the coherent state $|\alpha\rangle$. Hence the designation of $A(k)$ as a generalized destruction operator.

Despite the fact that Hermicity of the generators is lost when k fails to satisfy $k > -1$, we note that for negative integral k Hermiticity can be restored via a simple modification based on the fact that the $H_i(k)$ now reduce the oscillator space to two invariant subspaces. In the subspace defined on the number basis by $n \geqslant |k|$ we retain the definitions of eq. (5), while for $n<|k|$ we multiply the $H_{\pm}(k)$ by i. The modified $H_i(k)$ now satisfy the commutation rules of O_3 in the $|k|$-dimensional subspace and continue to satisfy those of $O_{2,1}$ for $n \geqslant |k|$. It may appear that we no longer have a well-defined set of states $|\alpha; k\rangle$ because of the singular nature of $A(k)$. However, inspection of eq. (4) shows that the $|\alpha; k\rangle$ remain well-defined for all real k, being entirely confined to the subspace $n<|k|$ when k is a negative integer. Furthermore, they remain eigenstates of $A(k)$ for all values of k indefinitely close to the negative integers. This suggests that we retain the concept of eigenstate of $A(k)$ even in the limit. Since the form of eq. (1) is unaffected by our modifications, we expect that the $H_i(k)$ still generate transformations of the limiting $|\alpha; k\rangle$ into themselves. This can be verified by direct calculations, and implies the correctness of the concept of eigenstates of $A(k)$ even in the limit of finite-dimensional support. In fact, the limiting states $|\alpha; k\rangle$ are, in the usual language of O_3, precisely all those obtained by rotating a fixed state whose angular momentum is maximum in some definite direction, *i.e.* a state $|j, m=j\rangle$ with $j=\frac{1}{2}(|k|-1)$. The task of extending these arguments to the subspace $n>|k|$ presents no new difficulties and will be treated in detail elsewhere.

Although our original motivation stemmed from dynamical considerations, our procedure introduces new group-theoretical techniques which give strong promise of general applicability. This generalization has so far proceded as follows:

1) Generalize eq. (1) to include the possibility of several commuting A-operators whose common, nondegenerate eigenstates span a Hilbert space. Thus we now seek a set of Hermitian generators H_a and operators A_l which satisfy

$$[H_a, A_l] = f_{al}(A_1, A_2, \ldots, A_R)\,. \tag{8}$$

The search for the H_a is aided by the fact that if they exist they must be expressible in the form

$$H_a = \sum_l f_{al}(A_1, \ldots, A_R)\frac{\partial}{\partial A_l} + g_a(A_1, \ldots, A_R)\,. \tag{9}$$

Our conjecture that we will thereby generate representations of successively higher-rank Lie algebras has so far been encouraged by work (to be published) on the cases $R=1, 2$. Furthermore, although no assumptions were made initially to the effect that the A_l are generalized destruction operators, our results have been expressible in terms of A_l with this property.

2) We note that $A(k)$ of eq. (3) is expressible in terms of the generators of eq. (5). This suggests a less ambitious procedure for more complicated situations, namely to assume in eq. (8) that the H_a are given and to seek A_l which are functions of these. Because the A_l would be functions of the generators, they would commute with the Casimir operators and leave any representation space invariant. Their eigenstates would provide representations of the Lie algebra. The following argument will indicate how such A_l may be found at least for those Lie algebras whose generators may be represented as bilinear forms in boson creation and destruction operators. For such cases we define the A_l as formal functions of the generators given by $A_l = a_1^{-1}a_l = (a_1^+ a_1)^{-1}a_1^+ a_l$. Since the a_l transform linearly among themselves under the action of the group, the A_l will undergo linear fractional transformations among themselves and thus satisfy our conditions. These ideas have been successfully applied to a number of cases, including O_4, $O_{3,1}$ and all rank-one algebras ([4]).

3) As the parameter k of the states $|\alpha; k\rangle$ of eq. (4) assumes negative values, we pass continuously through nonunitary representations of $O_{2,1}$ to successively higher-dimensional unitary representations of O_3. We conjecture that this is a manifestation of a general feature of the A_l-operators wherein we can find trajectories in a parameter space which take us continuously through nonunitary representations of noncompact groups to isolated points associated with unitary representations of compact groups. (This feature is reminiscent of the behaviour of Regge trajectories.)