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Quantum-Related Reference Frames and the Local Physical Significance of Potentials

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In a sequel to our previous paper we discuss two thought experiments which show that potentials in force-free regions have not only a nonlocal physically measurable significance (via, e.g., $\oint \mathbf{A} \cdot d\mathbf{l}$*), but, in singly connected portions of that region, also have a necessary local significance (via their quantum spread* $\Delta\mathbf{A}$*, which cannot be neglected). We then show, in continuation to the foregoing paper, how such* **A** *arise "geometrically" as kinematic quantities associated with the transformation between "quantum-related" reference frames, e.g., when the relative frame velocities are* q*-numbers possessing a quantum spread.*

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1. INTRODUCTION

In Ref. 1 attention was drawn to the physical (i.e., measurable) context of electromagnetic potentials $A_\mu(x)$ in quantum theory. The existence of nonzero A_μ in a *forceless* (rot $A_\mu = 0$) (multiply connected) region would, e.g., show up through a measurable shift of the interference pattern created by two branches of an electron wave packet.

It was emphasized,[1] however, that the measured effect depends on a *nonlocal* quantity associated with A_μ, namely on $\oint \mathbf{A} \cdot d\mathbf{l}$, whereas in any *local* portion of the region, **A** can be gauge-transformed to zero, without effecting the experimental result. Hence it could have been claimed that the physical effect of the A_μ is, by necessity, of nonlocal nature.

We here wish to show that, under suitable circumstances, the A_μ will necessarily possess a physical meaning in any local (even if forceless) region. As a corollary, the analysis of typical thought experiments will lead to the concept of "*quantum-related*" *frames of reference*, i.e., frames in which the distinguishing physical variable (e.g., the relative velocity of two frames) cannot always be "sharp" and must be considered as a q-number. It then further follows that one may *identify* the potentials with the *quantum* variables of the measuring laboratory; in particular, as kinematical q-numbers associated with the transformation of two moving frames of reference. These conclusions all constitute a natural, logical sequel to our results on the geometric meaning of potentials in a previous paper.[2]

2. A THOUGHT EXPERIMENT: ROTATING CHARGED CYLINDER

Imagine a very ("infinitely") long, rigid, hollow, homogeneously charged cylinder of radius R, perpendicular to the plane (Fig. 1), rotating clockwise with azimuthal velocity v_θ. There will be a magnetic flux ϕ inside the cylinder, but none outside. We may assume that the radial electric field outside has been shielded by a conducting grounded cylinder or, equivalently, by adding a concentric cylinder with opposite charge rotating in the opposite sense. Thus the outside region will be strictly forceless, but, of course, it must have

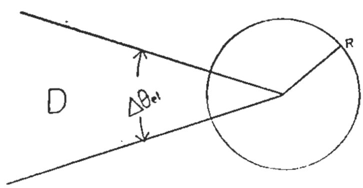

Fig. 1

$\mathbf{A}(r, \theta) \neq 0$, at least in some parts of the outside region, to give $\oint \mathbf{A} \cdot d\mathbf{l} = \phi$ on any circle enclosing the cylinder.

One may choose a gauge in which $\mathbf{A}(r, \theta)$, at any point r, θ outside, is simply proportional to the rotational velocity v_θ^{cyl} of the cylinder, and hence to its rotational momentum p_θ^{cyl}:

$$\frac{e}{c} A(r, \theta) = \frac{e}{c} \frac{\gamma}{r} v_\theta^{\text{cyl}} = \frac{e}{c} \frac{\gamma}{rIR^2} p_\theta^{\text{cyl}} = \frac{\alpha}{r} p_\theta^{\text{cyl}} \tag{1}$$

where γ is the charge density on the cylinder, I its moment of inertia, and R its radius, and $\alpha = e\gamma/cIR^2$.

Consider now an electron e moving in some local portion D of the outside region (see Fig. 1). If θ_{el}, p_θ^{el} denote its azimuthal canonical variables and θ_{cyl}, p_θ^{cyl} those of the cylinder (where θ_{cyl} measures the angular position of some mark on the cylinder), then the commutation relations give

$$(p_\theta^{\text{el}} - (\alpha/r)\, p_\theta^{\text{cyl}}, \theta_{\text{el}} + (r/\alpha)\, \theta_{\text{cyl}}) = 0 \tag{2}$$

However, from (1) the azimuthal velocity v^{el} of the electron is given by

$$mrv_\theta^{\text{el}} = p_\theta^{\text{el}} - (e/c)\, A = p_\theta^{\text{el}} - (\alpha/r)\, p_\theta^{\text{cyl}} \tag{3}$$

Equations (3) and (2) imply that v_{el} and $\theta_{\text{el}} + (r/\alpha)\, \theta_{\text{cyl}}$ can be sharp simultaneously. There is a correlation between θ_{el} and θ_{cyl} which may make $\theta_{\text{el}} + (r/\alpha)\, \theta_{\text{cyl}}$ much sharper than $\Delta\theta_{\text{el}}$.

But, on the other hand, the relations

$$(p_\theta^{\text{el}}, \theta_{\text{cyl}}) = 0, \qquad (p_\theta^{\text{el}}, \theta_{\text{el}}) = -i\hbar \tag{4}$$

imply

$$(p_\theta^{\text{el}}, \theta_{\text{el}} + (r/\alpha)\, \theta_{\text{cyl}}) = (p_{\text{el}}, \theta_{\text{el}}) = -i\hbar \neq 0 \tag{5}$$

so that $\Delta p_\theta^{\text{el}}\, \Delta(\theta_{\text{el}} + (r/\alpha)\, \theta_{\text{cyl}}) \geqslant h$, i.e., $\Delta p_\theta^{\text{el}}$ must be "very large" if $\theta_{\text{el}} + (r/\alpha)\, \theta_{\text{cyl}}$ is sharp. *This can be reconciled with a sharp v_{el} only if p_θ^{el} and v_{el} are related by a vector potential $A(\theta)$* [see (3)] with a correspondingly large uncertainty ΔA.

Thus some *local* presence of $A(\theta)$ (i.e., its quantum spread) is necessary and could not, e.g., be gauge-transformed away (as would still be permitted if only the nonlocal quantity $\oint \mathbf{A} \cdot d\mathbf{l}$ were to have physical meaning).

Thus the possibility of (in principle) measurable quantum correlations between a charged particle, confined to a force-free (multiply connected) region, and an electromagnetic source outside that region may be taken as an indication of the existence of a vector potential inside that region.

In our previous paper[2] we have seen that, in a similar geometric setup,

A can be further interpreted as the relative angular velocity of two mutually rotating frames (in the present case, the stationary frame in which the above discussion was described and the rotating frame in which the cylinder is at rest—in this case one would have to use a grounded shielding cylinder rather than a charged cylinder moving in the opposite sense in order to obtain a vanishing electric field outside). In the next section we shall discuss a simpler setup in which, again, the transition between two moving frames necessarily requires the introduction of a vector potential whose quantum spread is nonzero and which is simply related to the relative velocity between the two frames.

3. QUANTUM-RELATED GALILEAN FRAMES AND THEIR ASSOCIATED VECTOR POTENTIALS

Consider now a particle (electron, say) e, an inertial (infinitely heavy) frame of reference A, and a frame B (of large but finite mass m_B) moving with constant (nonrelativistic) velocity V relative to A. Let $x_{e/A}$, $x_{B/A}$, $p_{A/B}$, etc. denote the positions of e and B relative to A, the momentum of A relative to B, etc., and let $v_{B/A} = -v_{A/B} \equiv V$. All our considerations will be confined to the nonrelativistic regions.

We now wish to compare two alternative ways of transforming from the use of frame A to that of frame B as stations for observing e: (I) as a "simple" Galilean transformation of frames of reference, (II) as a transformation introducing a vector potential.

We shall see that, although classically both descriptions are equivalent, quantum mechanically, because of the finitness of m_B description, II will contain additional physical features which will make it preferable to I.

I. By a "simple" Galilean transformation we mean the transformation formula of momentum resulting from

$$x_{e/A} = x_{e/B} + V_{B/A}t = x_{e/A} + Vt \tag{6}$$

by differentiation and multiplication with m_e, i.e.,

$$p_{e/A} = p_{e/B} + m_e V \tag{7}$$

Transformations (6) and (7) can be achieved by a canonical transformation with the generator

$$S_I = (p_{e/A} - Vm_e)(x_{e/B} + Vt) \tag{8}$$

giving

$$x_{e/A} = \partial S_1/\partial p_{e/A} = x_{e/B} + Vt, \qquad p_{e/B} = \partial S_1/\partial x_{e/B} = p_{e/A} - m_e V$$

In this case the transformation of the Hamiltonian $H(x_{e/A}, p_{e/A}) = p_{e/A}^2/2m_e$ would give

$$H(x_{e/B}, p_{e/B}) = H(x_{e/A}, p_{e/A}) - (\partial S_1/\partial t)$$
$$= [(p_{e/B} + m_e V)^2/2m_e] - Vp_{e/B} = (p_{e/B}^2/2m_e) + \tfrac{1}{2}m_e V^2 \tag{9}$$

We shall now see that this "simple" description may lead to the following difficulty when applied to the quantum domain. The canonical commutation relations give

$$((1/m_e)\, p_{e/A} - (1/m_e)\, p_{B/A}, m_e x_{e/A} + m_B x_{B/A}) = 0 \tag{10}$$

Within the description I we have $p_{e/A} = m_e v_{e/A}$, $p_{B/A} = m_B v_{B/A}$, and $p_{e/B} = m_e v_{e/B}$. Hence (10) goes over into

$$(v_{e/A} - v_{B/A}, m_e x_{e/A} + m_B x_{B/A}) = 0$$

or, using (6), etc.,

$$(v_{e/B}, m_e x_{e/B} - (m_e + m_B)\, x_{A/B}) = 0$$

or (11)

$$(p_{e/B}, m_e x_{e/B} - (m_e + m_b)\, x_{A/B}) = 0$$

But $(x_{e/B}, p_{e/B}) = i\hbar$ and $(x_{A/B}, p_{e/B}) = 0$ imply

$$(p_{e/B}, m_e x_{e/B} - (m_e + m_B)\, x_{A/B}) = -im_e\hbar \neq 0 \tag{12}$$

which is inconsistent with (11). This inconsistency can be resolved by introducing a vector potential into the relation between $p_{e/B}$ and $v_{e/B}$.

II. This is achieved by an alternative canonical transformation,

$$S_{\text{II}} = p_{e/A}(x_{e/B} + Vt) \tag{13}$$

giving

$$x_{e/A} = \partial S_{\text{II}}/\partial p_{e/A} = x_{e/B} + Vt \tag{14}$$

(which is consistent with the classical definition (6) of our Galilean transformation), and

$$p_{e/B} = \partial S_{\text{II}}/\partial x_{e/B} = p_{e/A} \tag{15}$$

But

$$p_{e/A} = mv_{e/A} = mv_{e/B} + mV \tag{16}$$

[the first part of (16) follows from A being an *inertial* system in which we assume no magnetic fields; the second part follows from differentiation of (14)]; hence

$$p_{e/B} = m_e v_{e/B} + A_e \tag{17}$$

where the "*vector potential*" A_e *has the simple kinematic meaning*

$$A_e = m_e V = m_e \times \text{(relative frame velocities)} \tag{18}$$

Alternatively, this result follows from the change in the Hamiltonian

$$\begin{aligned} H(x_{e/B}, p_{e/B}) &= H(x_{e/A}, p_{e/A}) - (\partial S_{11}/\partial t) = (p_{e/B}^2/2m_e) + V p_{e/B} \\ &= [(p_{e/B} + m_e V)^2/2m_e] - \tfrac{1}{2} m_e V^2 \end{aligned} \tag{19}$$

We now return to the vanishing commutator (10) and again replace $p_{e/A}$, etc., by $p_{e/B}$, etc., now using (17) and (18). We obtain

$$(v_{e/B}, m_e x_{e/B} - (m_e + m_B)\, x_{A/B}) = 0$$

or

$$(p_{e/B} - m_e V, m_e x_{e/B} - (m_e + m_B)\, x_{A/B}) = 0 \tag{20}$$

whereas

$$(p_{e/B}, m_e x_{e/B} - (m_e + m_B)\, x_{A/B}) = -i m_e \hbar \tag{21}$$

as before. Now (21) and (20) are consistent by virtue of (16), the relation $p_{e/A} = m_e v_{e/A}$, and the canonical commutation relations.

Thus if $v_{e/B}$ and $m_e x_{e/B} - (m_e + m_B)\, x_{A/B}$ are chosen sharp simultaneously [as permitted by (20)], (21) implies that $A_e = m_e V$ must by necessity remain unsharp.

Thus as in II, the "vector potential" cannot be thought of as a pure mathematical, "auxiliary" quantity (which could perhaps be gauge-transformed away locally), but manifests itself as a locally necessary physical quantity (at least its quantum spread).

It should be emphasized that *classically* the description I is indeed correct and consistent, in spite of the fact that one of the "Galilean" frames has a finite mass (m_B). Upon transition to quantum theory, however, the transformation from a description of the electron *e* from the viewpoint A to the description from the viewpoint B necessitates taking into account also the "kickback" which the frame B of finite mass receives from the observation of *e*. This "kickback" (which can be neglected classically) introduces an uncertainty in the relative velocity V of the two frames. Thus the parameter of transformation, here V, must be considered as a *q*-number; the two reference frames are "*quantum related*."

Beyond these conclusions (which parallel those of II), the results just obtained suggest a more far-reaching consequence, namely that—within the framework of quantum theory—potentials can be interpreted as kinematical variables associated with a change of frame of reference and, vice versa, a change of frames will show up in the appearance of potentials.

REFERENCES

1. Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
2. Y. Aharonov and G. Carmi, *Found. Phys.* **3**, 493 (1973).