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Measurement of Noncanonical Variables

Y. AHARONOV* AND JOHN L. SAFKO

Department of Physics and Astronomy, University of South Carolina Columbia, South Carolina 29208

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We consider the problems of measurements of noncanonical variables such as velocity, kinetic energy, and the electric field in the quantum domain. We show that there is an essential difference between such measurements and the measurements of canonical variables, i.e., noncanonical observables must be changed, in an uncontrollable way, while being measured. We then construct a formal theory for the measurement of noncanonical variables. We apply this theory to measurements of velocity, kinetic energy, and the electric field and show how it clarifies and simplifies previous discussions of these measurements.

I. THE PROBLEM OF MEASUREMENT

Ever since its foundation measurement theory has been one of the more controversial subjects of quantum theory. Classical mechanics has a relatively simple measurement theory since the interaction between the measurement apparatus and the system being examined can be made as small as desired. This cannot be done in quantum theory. Thus in classical physics the system can be left undisturbed by the measurement while quantum mechanical systems must be disturbed in some fashion by a measurement.

There are a number of compendiums on the work that has been done on quantum mechanical measurement theories [1, 2, 3, 4]. We concern ourselves here only with the question as to whether any one dynamical variable may be measured as accurately as desired, without its being disturbed.

Quantum as well as classical observables may be divided into two classes: canonical and noncanonical. A canonical observable is one defined solely in terms of the momenta and positions of a given system (and possibly time). There are other observables of physical interest which are noncanonical. To see this, consider any canonical operator A. Its time derivative $\dot{A}$ is given by

$$\dot{A} = (1/i\hbar)[A, H] + \partial A/\partial t,$$

* Permanent address: Physics Department, Tel-Aviv University, Israel, and Physics Department, University of South Carolina, Columbia, SC 29208.

where H is the Hamiltonian of the system. $\dot{A}$ is a noncanonical observable since its definition depends upon the Hamiltonian of the system. As we will see later, if the time derivative $\dot{A}$ does not commute with the operator A itself, then there will be an important distinction in its measurement theory from the measurement theory of canonical variables. Examples of noncanonical variables of physical interest include velocities, kinetic energies and electric fields.

In the next section we will formulate a measurement theory of canonical variables in the impulsive limit. By impulsive we mean the limit in which the time duration of the measurement approaches zero. Section III will extend these considerations to the impulsive measurement of noncanonical variables where we find that different considerations must be made. We then extend this measurement theory to non-impulsive measurements. Finally, in Section V we consider three applications, two of these are results of special interest: (a) a clarification of the confusion surrounding the interpretation of the time–energy uncertainty principle [5, 6]; (b) an interpretation and simplification of the Bohr–Rosenfeld analysis of the measurement of electromagnetic fields [7].

II. Impulsive Canonical Measurement Formalism

The canonical formalism of quantum measurement theory to be presented here is an outgrowth of the early work of von Neumann [8]. Von Neumann's view was that, even when we are discussing the quantum domain, there should be a way to describe the combined system of measuring device and measured system according to quantum rules. We assume that one can think about all the degrees of freedom of the measuring device, together with the degrees of freedom of the system under investigation, as one extended system which has some Hamiltonian, H, to describe it. Qualitatively we have three time periods: At first there is no interaction and $H = H_S + H_{MD}$ where H_S is the Hamiltonian of the system and H_{MD} is the Hamiltonian of the measuring device. This is followed by a period of time, T, where the system and measuring device interact. During this period

$$H = H_S + H_{MD} + H_i, \tag{1}$$

where H_i is an interaction Hamiltonian. Finally the interaction is turned off and $H = H_S + H_{MD}$ once again.

Usually the following four criteria for the measurement of a variable A_s which is a Hermitian operator representing a physical observable of the system are taken:

(1) For simplicity, measurements should be made in as short a time, T, as possible. If the limit $T \to 0$ can be taken, the measurement is termed *im-*

pulsive. Impulsive measurements are useful for the following reason. If A_s is not a constant of the motion it will change during the measuring interval and thus the measured quantity may be disturbed by an uncertain amount due to the interaction with the apparatus. For example, if we are measuring the position of a moving particle, its momentum will be changed in an indeterminate fashion during the measurement and hence the average position is effected by the measurement process. For the remainder of this section we will take the impulsive limit, coming back to the effects of a finite duration of measurement in Section IV. We do this to make it easier to compare the essential difference in canonical and noncanonical measurement theory.

(2) The measurement must not change the measured variable A_s. This can be assured if $[A_s, H_i] = 0$ provided the measurement is impulsive.

(3) When the measurement is completed there is a change in the measuring device which corresponds to the value of A_s.

(4) Finally we want a magnification so the reading in the measuring device will be stable. Namely, a microscopic change in the quantity to be measured will produce a macroscopic change in the measuring device.

These requirements can be satisfied in nonrelativistic quantum theory by an interaction Hamiltonian containing terms like $g(t)\, A_s B_{\mathrm{MD}}$ where B_{MD} is some variable in the measuring device and $g(t) = 0$ before and after the measurement. Thus conditions 2 and 3 are met as will be shown later. Since only integrals of $g(t)$ appear, we will consider

$$g(t) = \begin{cases} g_0, & 0 < t < T, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Condition 1 for impulsive measurements is satisfied by letting T approach zero while keeping the product $g_0 T$ finite. Choosing the product $g_0 T$ sufficiently large will also satisfy condition 4. Thus all our requirements for a measurement are met. The particular form of $g(t)$ given in Eq. (2) is not essential for the general conclusions. We only need that $\int_0^T g(t)\, dt$ be finite and large as $T \to 0$.

Before we proceed let us note one important difference between classical and quantum measurements. Consider a general observable, $f(x, p)$. Classically we may measure x and then p, or conversely, and finally calculate f. Quantum mechanically this cannot be done, since the measurement of x will change p and therefore f. Hence, quantum mechanics forces us to find a completely different physical arrangement for each different observable.

It turns out that the essential elements we wish to emphasize in our discussion here can be fully illustrated by the very simple example of the measuring device as well as the measured system being represented by free particles before and after the measurement. Thus we take

$$H_S = p^2/2m, \qquad H_{MD} = \Pi^2/2M$$

and (3)

$$H_i = g(t)\, A_s B_{MD}\,,$$

where M/m is extremely large, Consider first the choice $A_x = x$, the position of the measured particle, and $B_{MD} = y$, the position of the measuring device. Thus,

$$H_i = g(t)\, xy,$$

then, working in the Heisenberg representation (as we will do for the remainder of the paper),

$$\dot{x} = p/m,$$

$$\dot{y} = \Pi/M,$$

$$\dot{p} = -gy,$$

and

$$\dot{\Pi} = -gx.$$

For M sufficiently large y can be considered constant during the interaction. We call this constant y_0. So for $0 < t < T$ we have

$$x = x_0 + (p_0/m)\, t - \tfrac{1}{2} g_0\, y_0 t^2 \tag{4}$$

and

$$\Delta\Pi = \Pi(t) - \Pi(0) = -g_0 \int_0^T x\, dt = -g_0 T\bar{x} + \tfrac{1}{6} g_0{}^2 T^3 y_0\,, \tag{5}$$

where

$$\bar{x} = x(0) + p(0)\, T/2m \tag{6a}$$

or

$$\bar{x} = x(T) - p(T)\, T/2m. \tag{6b}$$

We now consider our impulsive limit $T \to 0$, $g_0 \to \infty$ taken such that

$$G = g_0 T \tag{7}$$

is finite.

So Eq. (5) can be rewritten as

$$\Pi(0)/G = (\Pi(T)/G + \bar{x}) = A(T), \tag{5'}$$

where in the impulsive limit

$$\bar{x} = x(0) = x(T). \tag{8}$$

Since G is to be taken large, Eqs. (5) and (8) clearly satisfy conditions 2 and 3 of our requirements for a good measurement.

Equation (5′) may be interpretated in the following way. Let us call Π the macroscopic "indicator" of the measuring device. We start our measurement at $t = 0$ with this indicator being at a reasonably definite macroscopic value Π_0 with a finite uncertainty $\Delta\Pi_0$. By choosing G to be sufficiently large we have $\Delta\Pi_0/G$ as small as desired. That means that the uncertainty in $A(T)$ may be made as small as desired. Thus when the measurement is finished, a complete correlation between the indicator value $\Pi(T)$ and the measured quantity $\bar{x}$ has been established. Thus a macroscopic measurement of $\Pi(T)$ will give $\bar{x}$ as precisely as desired.

It may be worthwhile here to complete the above discussion by considering the wavefunction respresenting the system plus measuring device. Since we work in the Heisenberg representation, this wavefunction remains the same throughout the measurement until the "collapse" of the wavefunction takes place when we measure $\Pi(T)$. Let this wavefunction be

$$\Psi(x, y) = e^{i\Pi_0 y}\Phi(x),$$

where $\Phi(x)$ represents the initial state of the observed system and where for simplicity we took the measuring device to be in an eigenstate of Π with an eigenvalue Π_0. After time T the same wavefunction represents an eigenstate of the operator $A(T)$. Hence a measurement of $\Pi(T)$ will "collapse" Ψ to a product of the form

$$\Psi'(x, y, T) = e^{i\Pi(T)y}\,\Phi'(x),$$

where $\Phi'(x)$ is an eigenstate of the operator

$$\bar{x} = x(T) - p(T)\,T/2m = x(T) = x(0)$$

as expected in the impulsive limit.

A similar analysis shows that momentum or any function of position and momentum can be impulsively measured by the same approach. We see then, that in the nonrelativistic domain, where we are free in principle to consider arbitrary interactions, we can impulsively measure any canonical variable as precisely as desired. Before proceeding to noncanonical variables, however, the following two points should be clarified.

(1) Although the Hamiltonians considered so far appear to be noninvariant both under space and time translations, it should be remembered that the positions and time appearing in the Hamiltonian are relative quantitites. Thus in Eqs. (3) the x and y, for example, stand for $(x - z)$ and $(y - z)$ where z is the position of some very massive reference object in the laboratory. This reference object remains stationary throughout the experiment. It is straight-

forward to show that enlarging the system under consideration to include the z degree of freedom as well will add no new information provided the quantum fluctuations of z can be neglected, as indeed is the case when the reference object is sufficiently heavy. Including the z degree of freedom makes the Hamiltonian invariant under translations without affecting in any essential way the preceeding agreement.

(2) In our discussion we considered the measurement of a quantity (position) which does not commute with an additive conserved quantity (momentum). Thus it seems that the objections of Wigner [9], Yanse and Araki [10] concerning such measurements should apply. However, in our example we assume that the mass of the measuring device is macroscopic; and therefore, their objections do not apply.

III. The Impulsive Measurement of Noncanonical Variables

Let us now consider a simple example of noncanonical variable: the velocity of a free particle. From the results of Section II we conclude that the momentum of a free particle can be measured without changing its value. It is easy to show that the velocity cannot be measured without changing its value. Suppose at some time, say $t = 0$, we know the position,

$$x(0) = x_0 \pm \Delta x(0),$$

and velocity

$$v_x(0) = v_0 \pm \Delta v(0),$$

of a particle where

$$\Delta x(0) \Delta v_x(0) \geqslant \hbar/m.$$

At some later time, T,

$$x(T) = x_0 + v_0 T \pm \Delta x(T)$$

where

$$\Delta x(T) = \Delta x(0) + \Delta v_x(0)\, T.$$

If T is small, we can neglect the $\Delta v_x(0)\, T$ term. We now measure v_x exactly so that $\Delta v_x(T) = 0$. If this measurement takes a time $\Delta T \ll T$, then

$$\Delta x(T + \Delta T) = \Delta x(T) + \Delta \int_T^{T+\Delta T} v_x(T')\, dt', \tag{9}$$

where the last term can be neglected since we are assuming that v_x is unchanged by the measurement. Then,

$$\Delta x(T + \Delta T) \simeq \Delta x(T), \tag{9'}$$

so

$$x(T + \Delta T)\,\Delta v_x(T + \Delta T) \to 0.$$

This violates the uncertainty relations, so the velocity must have been changing during the interval ΔT. If so then the second term in Eq. (9) cannot be neglected and thus Eq. (9′) is incorrect.

To trace the source of this difficulty let us try to apply our formalism to the problem of noncanonical measurement. The most general noncanonical variable can be written as a function $f(q, p, \dot{q}, \dot{p}, t)$. Without any loss of generality we will restrict ourselves to the simple case of noncanonical variables which are the time derivatives of an arbitrary canonical variable $A(q, p)$. By a suitable canonical transformation we can make A a coordinate Q of the transformed system with conjugate momentum P. Hence the problem of measurement of a noncanonical variable is the same as that of the measurement of the (generalized) velocity. Quantum mechanically, the same arguments hold except unitary transformations are used instead of canonical transformations.

Since we can not directly put a velocity into the Hamiltonian we must use its canonical equivalence

$$\dot{Q} = -i\hbar[Q, H] = \partial H/\partial P.$$

Then the total Hamiltonian, H, is given by

$$H = H_S + H_{MD} + g(t)[\partial H/\partial P]\,B, \tag{10}$$

where we are using the notation of Eqs. (1)–(3). Note that what appears in the square bracket is $\partial H/\partial P$ not $\partial H_S/\partial P$.

We must now distinguish between two cases: $[\dot{Q}, Q] = 0$ or $[\dot{Q}, Q] \neq 0$. It is easy to show that this distinction is invariant under unitary transformations. For the case $[\dot{Q}, Q] = 0$, it is obvious that the Hamiltonian, H_S, of the system can not be quadratic or higher in the momentum P conjugate to Q. Hence the system Hamiltonian must be $H_S = Pf_1(Q, Q',..., Q'', P',..., P'') + f_2(Q, Q',..., Q'', P',..., P'')$ where $Q',..., Q''$ are the other coordinates of the system and $P',..., P''$ their conjugate momentum. If H_S is at most linear in P then and only then is $\partial H/\partial P = \partial H_S/\partial P$. Thus we satisfy condition 2 of Section I. We can show this by differentiating Eq. (10) with respect to P giving

$$\partial H/\partial P = \partial H_S/\partial P + g(t)(\partial^2 H/\partial P^2)\,B.$$

Clearly it is consistant to choose $\partial H/\partial P = \partial H_S/\partial P$ if H_S is linear in P. On the other hand if $\partial H/\partial P = \partial H_S/\partial P$, then $g(t)\,(\partial^2 H_S/\partial P^2)\,B = 0$ which is inconsistent unless H_S is linear in P.

Suppose we now consider the case $[\dot{Q}, Q] \neq 0$. Since $\dot{Q} = \partial H/\partial P$ and since for $[\dot{Q}, Q] \neq 0$, $\partial H/\partial P \neq \partial H_S/\partial P$, $\dot{Q}$ must involve $g(t)$ and hence $\dot{Q}$ is not a constant of the motion even though it commutes with H. We will in the remainder of this section show that the correct approach is still to replace $\partial H/\partial P$ in Eq. (10) with $\partial H_S/\partial P$. We will see that this change in the content of Eq. (10) will enable us to measure $\dot{Q}$, but during the measurement process there will be compensating forces acting on the system and the measuring device that are not obvious from Eq. (10).

To understand what is occurring we go to the Lagrangian formalism considering the particular example of the measurement of the velocity of a free particle. The Lagrangian formalism is chosen since it is formulated in terms of noncanonical variables, but at the end we will go back to the canonical formalism which is more directly suitable in quantum mechanics.

We take the position of the particle to be x and use a measuring device which is a free particle of position y. Then the Lagrangian is

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\dot{y}^2 + g(t)\,\dot{x}y, \tag{11}$$

where $g(t)$ is as given in Eq. (2) and the measurement will be made impulsively.

This gives us equations of motion

$$d/dt(M\dot{y}) = -g(t)\,\dot{x} \tag{12a}$$

and

$$d/dt(m\dot{x} + g(t)\,y) = 0. \tag{12b}$$

Equation (12a) is of the desired form, since the change in the measuring device is proportional to the velocity of the measured particle. Equation (12b), on the other hand, shows us that it is not the velocity of the measured particle which is a constant of the motion, but the sum of the velocity and a term depending upon the position of the measuring device. This is unacceptable for quantum mechanics since we must measure the change in the momentum of the measuring device which means its position is uncertain. However a modification of Eq. (11) to the form

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\dot{y}^2 + g(t)\,\dot{x}y + g^2(t)\,y^2/2m \tag{11'}$$

leaves Eq. (12a) unchanged but changes Eq. (12b) to read

$$d/dt(M\dot{y}) = -g(t)(\dot{x} + g(t)\,y/m). \tag{12b'}$$

The measuring device now responds to a constant of the motion, so it measures the velocity without the effects of the measurement process. So we may compensate for the necessary change in the velocity of the particle by adding an extra force.

It turns out that the canonical formalism enables us to solve this seemingly complicated problem in a straightforward way. Just write the Hamiltonian which measures the momentum p of the particle. This is

$$H = p^2/2m + \Pi^2/2M - g(t)\,yp/m, \tag{13}$$

which gives us equations of motion

$$v = \dot{x} = p/m - g(t)\,y/m,$$
$$\dot{\Pi} = +g(t)\,p/m.$$

These equations are equivalent to Eqs. (12a) and (12b′) so by measuring the value of p/m we obtain the value of the velocity before or after the measurement; but of necessity, the velocity is uncertain during the measurement. Thus the Hamiltonian of Eq. (13) which can be rewritten as

$$H = (p - g(t)\,y)^2/2m - g^2(t)\,y^2/2m + \Pi^2/2M \tag{13′}$$

is the same as the one derived from the Lagrangian of Eq. (11′).

Although the Hamiltonian Eq. (13) or (13′) is most useful for quantum formalism, it is the Lagrangian which gives us physical insight as to the experimental arrangement necessary to measure velocity. Two interactions are needed; one is a coupling of the measuring device and system being studied similar to the coupling for canonical variables, while the other corresponds to a "spring" attached to the measuring device from some reference frame. It is interesting that these two together are the same as a single coupling to the canonical variable corresponding to the noncanonical one in the absence of the measurement.

In summary we conclude that although we cannot measure noncanonical variables without changing them during the measurement process we can arrange impulsive measurements so the following occur.

(a) The correct Hamiltonian for the measurement of a noncanonical variable $\dot{Q}$ is

$$H = H_S + H_{MD} + g(t)[\partial H_S/\partial P]\,B,$$

where P is the momentum conjugate to Q. In general this amounts to an interaction between the measuring device and system not only proportional to $\dot{Q}$ but also involving compensating forces.

(b) The value of the noncanonical variable after the measurement is completed is the same as it was initially. Unless this is arranged the measurement has not been completed. During the measurement the noncanonical variable is uncertain.

(c) The value of the noncanonical variable that the measuring device reads is the value before or after the measurement, not its value during the measurement.

The example of velocity in this section indicates the general prescription for writing the interaction for the measurement of noncanonical variables. (1) In the absence of the measurement each noncanonical variable has a canonical equivalent defined for the system (For example $v = p/m$). To measure a noncanonical variable, write an interaction Hamiltonian, with an interaction proportional to the above canonical variable. As we will see in the next section, if the measurement is not impulsive an extra term will be needed to correct for the time evolution. (2) Transform from the Hamiltonian to the Lagrangian formalism. This will tell us what compensating forces are needed in the laboratory to make the measurement.

IV. Nonimpulsive Measurements

The correct approach for nonimpulsive measurements can be found by analyzing the results of Section II. Consider Eq. (5). When the limit $T = 0$ cannot be taken there is an extra term proportional to $T^3 y_0$ present. This may be eliminated if we take the total Hamiltonian as

$$H = p^2/2m + \Pi^2/2m + g(t)\, xy + (\tfrac{1}{12})\, g^2(t)\, T^2 y^2, \tag{14}$$

where $g(t)$ is given by Eq. (2). We again take $g_0 T$ finite and large but do not go to the limit $T = 0$. Then the change in the momentum of the measuring device becomes

$$\Delta\Pi = \Pi(T) - \Pi(0) = -g_0 T \bar{x}, \tag{15}$$

even for finite times, T. If we let $G = g_0 T$ without a limiting process, then we obtain Eq. (5′) again and the analysis in the two paragraphs following Eq. (8) is applicable except that we no longer rigorously satisfy condition 2 since $\bar{x}$ is given by Eq. (6a) or (6b) rather than by Eq. (8).

Condition 2 must be modified so that it only applies in the impulsive limit of any finite duration measurement, since the uncertainty relations forbid a measurement of a system without some change in the system. We can guarantee that the change in the canonical quantity we are measuring is the minimum possible. In our example of the measurement of position Eq. (6) tells us that although $x(T) \neq x(0) + p(0)\, T/m$ its average value is the same as it would have been had we not made the measurement. If we had not added the term proportional to $g^2(t)\,T^2y^2$ in Eq. (14) then Eq. (15) would have been

$$[\Pi(0)/(g_0T) + (\tfrac{1}{6})\,g_0T^2y(0)] = [\Pi(T)/(g_0T) + \bar{x}]. \tag{6'}$$

Hence it would have been necessary to put the measuring device into a state for which the error in $(\Pi(0)/gT) + (\frac{1}{6})\,gT^2y(0))$ is small. This implies that y must be measured to a microscopic accuracy and hence the measuring device could not be considered macroscopic. So if the measurement is not impulsive, it is necessary to add compensating forces acting on the measuring device. In our example of a measurement of position, this corresponded to adding a spring attached to the measuring device. In general the corrections for nonimpulsive measurements can be provided by a term of the form $h(t, T)f(B_{MD})$ where the form of $h(t, T)$ is determined by $g(t)$.

The compensating forces needed to correct for the time evolution of both canonical and noncanonical variables can be found by considering the exact solution for small measurement times without this term and then adding what is needed as a correction. To measure noncanonical variables nonimpulsively we will need two types of compensating forces. The first of these will be like the "spring" needed in the impulsive measurement of velocity while the other will be needed to correct the time evolution of the system during the measurement.

V. Applications

In this section we consider three applications as examples of the formal theory given in Sections III and IV. The first of these is measurement of the kinetic energy of a free particle. The second is the measurement of the velocity of a harmonic oscillator. Finally, we consider the measurement of an electric field.

By the energy of a free particle we mean the noncanonical variable $\frac{1}{2}mv^2$. Before or after the measurement process this is the same as the Hamiltonian of that system. During the measurement, however, the interaction breaks this correspondence. If we measure velocity and calculate $E = \frac{1}{2}mv^2$ during the interaction, then the relevant uncertainty is Δv which from Eq. (13a) is given by

$$\Delta v = g(t)\,\Delta y/m. \tag{16}$$

Now the uncertainty introduced into x by our measurements will be

$$\Delta x = \Delta vT = g_0T\Delta y/m \geqslant g_0hT/(m\Delta\Pi). \tag{17}$$

Our only information on p comes from the measurement $\Pi(T) - \Pi(0) = -g_0Tp$.

So we know that

$$g_0 T \Delta p/m \geqslant \Delta \Pi. \tag{18}$$

Equations (7) and (8) guarantee that we satisfy the uncertainty relations for x and p. Since

$$\Delta E = v \Delta p \quad \text{and} \quad v \geqslant \Delta v,$$

we have

$$\Delta E\, T \geqslant \hbar, \tag{19}$$

where ΔE is the uncertainity in the energy of the system *during* the measurement interval $0 < t < T$. If some action is not taken to return the velocity to its original value, as we prescribed earlier, Eq, (19) would imply that the energy cannot be measured to an accuracy better than h/T. Thus we see that the confusion in the interpretation of the $\Delta E \Delta t$ uncertainity relation is related to the problem of measuring noncanonical variables. Since E must change during the measurement it was believed that this was a constraint on the measurement of energy [6]. Bohr originally arrived at the incorrect interpretation when he analyzed a collision example which did not return the velocity to its original value after a measurement [11]. He later realized as indicated in the Bohr–Rosenfeld paper on the measurability of the electromagnetic field that the change in the energy can be compensated [7]. But the connection between this problem and the general question of the measurement of noncanonical variables was not, as far as we can ascertain, recognized by him or anyone else.

We now examine the nonimpulsive measurement of the velocity of a harmonic oscillator. If we pick units such that the frequency, ω_0, and mass of the oscillator are unity, then we would, following our description of Section IV, write the impulsive Hamiltonian as

$$H = (p^2 + x^2)/2 + \Pi^2/2M + g(t)\,py \tag{20}$$

with p and x the harmonic oscillator's momentum and position and $g(t)$ as in Eq. (2). We are measuring the velocity of the oscillator by examining the change $\Delta \Pi$ in the momentum of a free particle of position y used as the measuring device. If the duration of the measurement is T and provided $M \gg g_0^2$,

$$x = x_0 \cos t + (p_0 + g_0 y) \sin t$$

and

$$p = (p_0 + g_0 y) \cos t - x_0 \sin t - g_0 y,$$

for $0 < t < T$ which gives us

$$\Delta \Pi = -g_0 \int_0^T p(t)\, dt = -g_0 T \bar{p} - g_0^2 y (\sin T - T),$$

where $\bar{p}$ is the average that $p(t)$ would have had in the interval 0 to T if we had not made the measurement. We can also use the identity $\bar{p} = \bar{v}$. Thus we see that instead of Eq. (20) we should have written

$$H = (p^2 + x^2)/2 + \Pi^2/2M + g(t)\,py - \tfrac{1}{2}\,g^2(t)\,y^2(\sin T - T)/T. \qquad (20')$$

For the Hamiltonian of Eq. (16′) the change in the momentum of the measuring device $\Delta\Pi$, is then given by

$$\Delta\Pi = -g_0 T\bar{v}.$$

If we consider the Lagrangian we find that in order to measure the velocity we need a compensating force which is a "spring," coupling the measuring device to the laboratory frame during the measurement, with a constant given by

$$k = -g_0{}^2\,(\sin T)/T.$$

As a final example of our formalism for the measurement of noncanonical variables, we consider the question of the measurability of the electromagnetic field strengths. The simplicity of our approach becomes obvious when we compare the result with the approach of Bohr and Rosenfeld [7].

Let us consider the question of the single measurement of the x-component of the electric field, $E_x = -(\partial\phi/\partial x) - (1/c)\,\partial\mathbf{A}/\partial t$ where ϕ and $\mathbf{A}$ are the scalar and vector potentials, respectively. Since E_x involves $\partial A/\partial t$ it is a noncanonical variable. Following our prescription of the previous section we expect to obtain the correct Hamiltonian approach by coupling to the canonical analogue of E_x which is the x-component of the electromagnetic momentum, Π_x.

Our test body will be a dipole of typical dimensions L. It could be the two charged spheres considered by Bohr and Rosenfeld. Thus our total Hamiltonian is just the sum of H_{EM}, the electromagnetic Hamiltonian, H_{MD}, the measuring device Hamiltonian, and H_{int}, the Interaction Hamiltonian. We take

$$H_{\mathrm{EM}} = \tfrac{1}{2}\sum_\lambda (\Pi_\lambda{}^2 + \nu_\lambda Q_\lambda{}^2),$$

$$H_{\mathrm{MD}} = p_1{}^2/2Mc^2 + p_2{}^2/2\overline{M}c^2,$$

and

$$H_{\mathrm{int}} = x_1 \sum_\lambda g_\lambda \Pi_{\lambda x} - x_2 \sum_\lambda n_\lambda \Pi_{\lambda x} + H_T,$$

where Q_λ, Π_λ are the coordinates and momenta of the decomposed electromagnetic field and x_1, p_1, x_2, p_2 are the coordinates and momenta of the respective charged spheres making up the measuring device and g_λ and n_λ are the transforms of the charge distributions. We note that taking g_λ and n_λ appropriately large will make a macroscopic charge in the measuring device. This is the same as Bohr and Rosenfeld's requirement of a large charge. We also see that the mass of the measuring device will need to be large to keep it in the region being studied. H_T is the correction needed for nonimpulsive measurements. Since each mode of the electromagnetic field is a harmonic oscillator we know the form of H_T for each mode will be the same as in the preceeding example. We fix one of the spheres by taking the limit $\bar{M} \to \infty$ and $x_2 = 0$ and dropping the subscript 1. The equations of motion for the movable charged sphere are

$$\dot{p}_x = -c \sum_\lambda g_\lambda \Pi_{\lambda x},$$

$$\dot{Q}_{\lambda x} = \Pi_{\lambda x} + g_\lambda x,$$

while

$$\ddot{Q}_{\lambda x} = \dot{x} g_\lambda - \nu_\lambda^2 Q_{\lambda x}.$$

Thus in the impulsive limit, denoting $g_\lambda T$ by G_λ,

$$\Delta p_x = -c \sum_\lambda G_\lambda \Pi_{\lambda x},$$

and hence we measure the electromagnetic momentum, which is unchanged since $[\Pi_{\lambda x}, H] = 0$. The electric field is proportional to the electromagnetic momentum before and after the interaction, so we have measured the value of the electromagnetic field before and after the measurement. Choosing the set of g_λ large allows us to reduce the $\delta\Pi_{\lambda x}$ to the arbitrary accuracy consistent with our approximation.

Let us now look at the compensating forces. In this case the Lagrangian (in the $\bar{M} \to \infty$, $x_2 = 0$ limit) becomes

$$L = \tfrac{1}{2} \sum_\lambda (\dot{Q}_\lambda^2 - \nu_\lambda^2 Q_\lambda^2) + \tfrac{1}{2} m \dot{x}^2 - x \sum_\lambda g_\lambda \dot{Q}_\lambda + \tfrac{1}{2} x^2 \sum_\lambda g_\lambda^2.$$

Thus we see that there is need of a compensating "spring" with spring constant

$$k = -\tfrac{1}{2} \sum_\lambda g_\lambda^2. \tag{21}$$

If we choose to hold our dipole open for a finite time, T, an extra compensating force should be introduced as indicated before. Equation (21) becomes

$$k = -\tfrac{1}{2} \sum_\lambda g_\lambda^2 \frac{\sin \omega_\lambda T}{\omega_\lambda T}. \tag{21'}$$

This compensating force corresponds to the one introduced by Bohr and Rosenfeld.

From Eq. (21) we can also see what we mean by an impulsive measurement. For our measurement to be impulsive it is necessary that the charge in the free system during the measurement period T can be neglected. That condition is satisfied provided the frequencies, ω_λ, present satisfy $\omega_\lambda \ll 1/T$. We clearly do not couple to frequencies much greater than c/L. Thus if $L \gg cT$, we have an impulsive measurement.

The problem of two measurements of a single field average separated in space by a distance D is handled in the same fashion. Let

$$H = \tfrac{1}{2}(\Pi_\lambda^2 + \nu_\lambda^2 Q_\lambda^2) + p_1^2/(2Mc^2) + p_2^2/(2\overline{M}c^2) + x_1 \sum_\lambda g_\lambda \Pi_\lambda + x_2 \sum_\lambda n_\lambda \Pi_\lambda ,$$

where x_1 and x_2 are the displacements of each measuring device from its equilibrium. Then in the impulsive limit

$$\Delta p_1 = -c \sum_\lambda g_\lambda T_1 \Pi_{\lambda_x} + 2c\tau x_2 \sum_\lambda g_\lambda h_\lambda ,$$

and

$$\Delta p_2 = -c \sum_\lambda h_\lambda T_2 \Pi_{\lambda_x} + 2c\tau x_1 \sum g_\lambda h_\lambda ,$$

where T_1 and T_2 are the durations of the measurements at x_1 and x_2, respectively, and τ is the overlap time of the measurements. The first terms in each case are just the results that would have been obtained if the other measurement had not been made.

Then the Lagrangian is

$$L = \tfrac{1}{2} \sum_\lambda (\dot{Q}_\lambda^2 - \nu_\lambda^2 Q_\lambda^2) + \tfrac{1}{2}M\dot{x}_1^2 + \tfrac{1}{2}\overline{M}\dot{x}_2^2 - x_1 \sum_\lambda g_\lambda \dot{Q}_\lambda$$
$$- x_2 \sum n_\lambda \dot{Q}_\lambda + \tfrac{1}{2}x_1^2 \sum_\lambda g_\lambda^2 + \tfrac{1}{2}x_2^2 \sum_\lambda n_\lambda^2 + x_1 x_2 \sum_\lambda g_\lambda n_\lambda .$$

The last three terms can be rewritten as

$$- \tfrac{1}{2} \sum_\lambda (x_1 g_\lambda - x_2 n_\lambda)^2 + x_1^2 \sum_\lambda g_\lambda^2 + x_2^2 \sum_\lambda n_\lambda^2.$$

If there is some overlap of the g_λ's and the n_λ's, we have "springs" coupling each measuring device to the laboratory frame and a "spring" connecting them. This again is similar to the results of Bohr and Rosenfeld. If the two domains don't overlap, i.e., $\Sigma_\lambda g_\lambda n_\lambda = 0$, the spring that couples the two test particles is not needed. The more general case of two arbitrary domains is handled in a fashion similar to that of nonimpulsive measurements.

Thus we see that the canonical approach when extended to noncanonical variables and to nonimpulsive measurements has given us, in a simple fashion, results similar to that of Bohr and Rosenfeld. The coupling chosen to measure a noncanonical variable, the electric field, is a canonical coupling to the field momentum. In the laboratory this means adding the "springs" of Bohr and Rosenfeld; however, we found this as a direct result of the formalism rather than from a detailed analysis of the particular problem.

VII. Conclusion

In conclusion, we find that a canonical formalism based upon the ideas of von Neumann can be extended to the measurement of noncanonical variables. During the measurement process many noncanonical variables such as velocity, kinetic energy, electric field strength, etc., must become uncertain. This uncertainty is inversely proportional to the duration of the measurement. Nevertheless, the measurement may be so made that we obtain the value of the noncanonical variable after the measurement is completed and that this is its value before. Thus, we are able to measure noncanonical variables as precisely as desired. We also found why the time–energy uncertainty relation cannot be interpreted as referring to the uncertainty of kinetic energy and the time taken to measure that energy. Finally, the compensatory "springs" introduced by Bohr and Rosenfeld to measure the electric field were shown to be a direct consequence of the formalism of noncanonical and nonimpulsive measurement theory.

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