Models of Renormalization.

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The problem of renormalization is as old as quantum field theory, and its roots extend back into classical electrodynamics. Since the late 1940's we have learned to live with the divergences of field theory, but it is only lately that we have begun to look upon them as a blessing rather than a curse. From the seminal work of CALLAN (1), SYMANZIK (2) and especially WILSON (3) we have learned that an understanding of renormalization effects is crucial to the physics of ultraviolet and infra-red limits in field theory and of critical phenomena in statistical mechanics. Unfortunately, because of the inherent complexity of any system in which renormalization effects occur naturally, it is very difficult to find a simple model of renormalization that exhibits the essential physics. We would like to report on some models which do just that.

Our models are based on a familiar and intuitive physical idea: the « virtual cloud » picture of renormalization. In this picture renormalization effects occur because even free particles are surrounded by a cloud of virtual quanta that arise from their interaction with the vacuum. The high-momentum modes of this virtual cloud are « stuck » to the particle, and contribute to its observed mass. Under the action of soft external forces the particle moves as if free. The only effect of its cloud being the mass renormalization.

A simple mathematical model of this effect can be made with the following Hamiltonian (4):

$$(1)\qquad H = \frac{1}{2m_0}(\boldsymbol{p}-\lambda\boldsymbol{q})^2 + \frac{1}{2}(\boldsymbol{\pi}^2+\omega^2\boldsymbol{q}^2)\,, \qquad [q_i, \pi_j] = i\delta_{ij}\,, \qquad [x_i, p_j] = i\delta_{ij}\,.$$

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(1) C. G. CALLAN: *Phys. Rev. D*, **2**, 1541 (1970).

(2) K. SYMANZIK: *Comm. Math. Phys.*, **18**, 227 (1970).

(3) J. KOGUT and K. WILSON: *The renormalization group and the ε expansion*, *Phys. Rep.*, to be published.

(4) The alert reader will notice that the transformation

$$\frac{m_1x_1+m_2x_2}{m_1+m_2} = x - \frac{\lambda\pi}{m_0\Omega^2}\,, \qquad \left(\frac{p_1}{m_1}-\frac{p_2}{m_2}\right)\sqrt{\mu} = \Omega\left(q - \frac{\lambda p}{m_0+\Omega^2}\right), \qquad \pi = k(x_1-x_2)\,, \qquad p = p_1+p_2\,,$$

tials which can transfer arbitrarily large momentum). This extra stability, which is a consequence of the discreteness of quantum energy levels, will be important when we attempt to build a field theory based on our model.

Another aspect of field-theoretic renormalization theory that is mirrored in our model can be seen by examining the equal-time commutator of the particle's position and velocity (⁷). The solution of the Heisenberg equations of motion for the particle is

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{p}}{m_0} - \frac{\lambda \boldsymbol{q}(t)}{m_0} = \frac{\boldsymbol{p}}{m_0} - \frac{\lambda}{m_0}\left\{\cos \Omega t \left(\boldsymbol{q}(0) - \frac{\lambda \boldsymbol{p}}{m\omega^2}\right) + \frac{\sin \Omega t}{\Omega}\,\boldsymbol{\pi}(0)\right\} - \frac{\lambda^2}{m_0}\,\frac{\boldsymbol{p}}{m\omega^2}\,. \tag{8}$$

The canonical commutation relations imply that the equal-time commutator of x_i and x_j is simply $i\delta_{ij}/m_0$. However, if we now smear both operators by averaging over a time τ large compared to the period of the oscillator, then we find that the rapidly oscillating terms in x average to zero and

$$\langle \dot{\boldsymbol{x}}(t)\rangle \equiv \frac{1}{\tau}\int\limits_{t-\tau/2}^{t+\tau/2} \dot{\boldsymbol{x}}(t')\,\mathrm{d}t' = \frac{\boldsymbol{p}}{m_0}\left(1 - \frac{\lambda^2}{m\omega^2}\right) = \frac{\boldsymbol{p}}{m}\,, \tag{9}$$

so that

$$\left[\langle x_i(t)\rangle, \langle \dot{x}_j(t)\rangle\right] = \frac{i}{m}\,\delta_{ij}\,, \tag{10}$$

since $[x_i(t), p_j] = i$ for all t. Equation (10) is the analogue in our model of a well-known fact of field theory: the smeared renormalized fields are finite operators even though the equal-time commutators of the fields themselves are singular. (In fact, although we have treated the transition $m_0 \to m$ as a mass renormalization, it is also a wave function renormalization if our model is considered as a zero-dimensional theory.)

Amusingly, our model appears to be the only one of a large class of Hamiltonians that can exhibit the effects we have described. Any Hamiltonian of the form

$$\frac{p^2}{2m_0} + f(q)p + g(q) + \frac{\pi^2}{2}$$

that satisfies the conditions A), B) and C) must in fact have the form (1).

The model we have described is nonrelativistic, but it is easy to generalize it. We write the invariant squared mass of a system as

$$m_0^2 = (p_\mu - \lambda q_\mu)^2 + (\pi_\mu \pi^\mu + \omega^2 q_\mu q^\mu)\,. \tag{11}$$

We use m_0 to generate proper-time equations of motion, and results exactly analogous to (1), (2) and (3) above (with the Galilean group replaced by the Poincaré group) follow immediately. In order to avoid imaginary masses we have to constrain the system. A simple constraint is $\pi_\mu \pi^\mu = 0$, and we have checked that there is a subset of solutions of the equations of motion that satisfies the constraint. This is the class of all solutions with $\lambda p_\mu - (\lambda^2 + \omega^2) q_\mu$ a lightlike vector. In particular, it contains all the solutions with constant velocity for the particle, since these satisfy $\lambda p_\mu - (\lambda^2 + \omega^2) q_\mu = 0$. The problem of incorporating this constraint into a quantum theory will be studied in a future work.

There are several interesting questions which we hope to investigate in further work on our model. Firstly, since the model is exactly solvable, we can compare renormalization of the exact theory with perturbative renormalization, and investigate the validity of the latter procedure. We can also study the effects of renormalization on the stability of the theory. For $\lambda/\omega \to \infty$ as the requirement that the physical mass be finite implies that the bare mass is $-\infty$. Does this mean that the theory is unstable?

Finally, we intend to construct a Fock space and a field theory using our model as a single-particle Hilbert space. (Remember that the model contains a representation of the Galilean group even for finite ω.) New features should arise because of the intrinsic nonlocality of any interaction which does not excite the particle's cloud.