Nonlinear vector product to describe rotations

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A method is developed for treating successive rotations in terms of nonlinear products of rotation vectors. Combinations of ordinary scalar and vector products yield the same results that may be more commonly arrived at by the use of quaternions or 2×2 unitary matrices. The method automatically provides the direction of the axis and the value of the rotation angle of the single rotation that is equivalent to the product of two or more successive rotations. This information is not readily obtained from the usual matrix rotation method of solving the successive rotation problem. The method is applied to the case of a rotation about the [1 1 1] direction, successive π rotations about orthogonal axes, and the treatment of continuous rotations.

INTRODUCTION

For many years it has been known that the single rotation corresponding to several successive rotations can be determined in various ways such as (a) the product of 3×3 real orthogonal matrices (group O-3), (b) the product of 2×2 unitary matrices (group SU-2), and (c) the product of quaternions. Of these three methods the most widely used and the only one which is of importance in elementary physics courses is the first. Unfortunately, there are some important applications in which this O-3 method is not the most useful; in particular, it is least useful for answering the question "What is the direction and rotation angle corresponding to the resultant of several successive rotations?" For example, Landau and Lifshitz[1] describe the product of rotations through an angle π about two axes intersecting at an angle θ as shown in Fig. 1, and even in this simple case their description is quite complicated.

We feel that the SU-2 and quaternion[2] approaches are not commonly presented in standard introductory graduate courses because the mathematical concepts that are involved are somewhat sophisticated. In this article we show how ordinary vector and scalar products can be used to reproduce the results of these two other methods.

The customary way to treat arbitrary rotations in three-dimensional coordinate space is to make use of a 3×3 rotation matrix R,

$$R=\begin{bmatrix} R_{11} & R_{12} & R_{13}\\ R_{21} & R_{22} & R_{23}\\ R_{31} & R_{32} & R_{33}\end{bmatrix}. \tag{1}$$

To describe the single rotation R_3 that is equivalent to two successive rotations R_1 and R_2 about arbitrary axes, we carry out a matrix multiplication

$$R_3=R_2R_1. \tag{2}$$

The matrices R_1 and R_2 are written for rotations around the coordinate axes, and it is not easy to tell the direction of the equivalent single axis of rotation or the angle about this single axis from an analysis of the form of R_3.

The purpose of the present article is to present a method of treating successive rotations in terms of simple nonlinear products. This method automatically provides the direction of the axis and the angle of the single rotation that is generated by two or more successive rotations about arbitrary axes.

THE NONLINEAR VECTOR PRODUCT

A rotation in three-dimensional space through an angle ϕ about a direction defined by the unit vector $\hat{n}$ may be characterized by a rotation vector $\mathbf{V}$:

$$\mathbf{V}=\hat{n}\sin(\phi/2). \tag{3}$$

We can also define a scalar f,

$$f=(1-V^2)^{1/2} \tag{4}$$

$$=\cos(\phi/2), \tag{5}$$

associated with this rotation. Equations (3) and (4) resemble the components of a quaternion.

A rotation characterized by the rotation vector $\mathbf{V}_1$ followed by a rotation characterized by the rotation vector $\mathbf{V}_2$ is given by the rotation vector $\mathbf{V}_3$. We show in the Appendix that $\mathbf{V}_3$ has the form of Eq. (A.14):

$$\mathbf{V}_3=\mathbf{V}_1\times\mathbf{V}_2+\mathbf{V}_1f_2+\mathbf{V}_2f_1. \tag{6}$$

We will call this nonlinear vector product operation the square cross product with the symbol

$$\mathbf{V}_3=\mathbf{V}_2\boxtimes\mathbf{V}_1. \tag{7}$$

We also show in the Appendix [Eq. (A.12)] that f_3 corresponding to the rotation vector $\mathbf{V}_3$ is obtained from the product

$$f_3=f_1f_2-\mathbf{V}_1\cdot\mathbf{V}_2. \tag{8}$$

This product will be called the square dot product with the symbol

$$f_3=\mathbf{V}_2\boxdot\mathbf{V}_1. \tag{9}$$

Thus we define the two vector products as

$$\mathbf{V}_2\boxtimes\mathbf{V}_1=\mathbf{V}_1\times\mathbf{V}_2+\mathbf{V}_1f_2+\mathbf{V}_2f_1 \tag{10}$$

and

$$\mathbf{V}_2\boxdot\mathbf{V}_1=f_1f_2-\mathbf{V}_1\cdot\mathbf{V}_2. \tag{11}$$

We can see from the forms of Eqs. (10) and (11) that the square cross product operation is not commutative,

$$\mathbf{V}_2\boxtimes\mathbf{V}_1\neq\mathbf{V}_1\boxtimes\mathbf{V}_2, \tag{12}$$

$$\frac{4J_e'}{\beta^{1/2}c} = (\alpha_2 - \alpha_1) + \left[\alpha_1\left(\frac{\beta}{\beta_1}\right)^{1/2} - \alpha_2\left(\frac{\beta}{\beta_2}\right)^{1/2}\right] - w'\frac{I_5}{I_4}\left[6 - \frac{3(\alpha_1+\alpha_2)}{2}\right] - w'\pi^{1/2} \times\left[\alpha_1\left(\frac{\beta}{\beta_1}\right)^{1/2} + \alpha_2\left(\frac{\beta}{\beta_2}\right)^{1/2}\right] + O(w'^2). \quad (23)$$

Relations (22) and (23) can be solved numerically to yield the leaf velocity w. However, one finds that, if the nondimensional photon energy flux is weak—specifically, if

$$J_e'/\alpha_1 \ll 1 \quad (24)$$

—then velocity w satisfies requirement (3).

Indeed, in the assumption of validity of relation (24),

$$(\beta/\beta_1)^{1/2} = 1 - (\pi^{1/2}/8)w' + (\pi^{1/2}/\alpha_1)J_e' + O(w'^2) + O(w'J_e'), \quad (25a)$$

$$(\beta/\beta_2)^{1/2} = 1 + (\pi^{1/2}/8)w' + O(w'^2), \quad (25b)$$

and substituting into (23)

$$w' \simeq \left(\frac{\pi}{16}\right)\left(\frac{J_e'}{1-(\alpha_1+\alpha_2)\gamma}\right)\left(1 - \frac{4}{(\pi\beta^1)^{1/2}c}\right) \simeq \left(\frac{\pi}{16}\right)\left(\frac{J_e'}{1-(\alpha_1+\alpha_2)\gamma}\right), \quad (26)$$

where

$$\gamma = (\pi/16)(4/\pi - 9/8) \simeq 0.029\ldots \quad (27)$$

In dimensional notation, employing (3) and (21):

$$w \simeq (\pi/16)[1-(\alpha_1+\alpha_2)\gamma]^{-1}(J_e/2KTn). \quad (28)$$

Here one should remember that approximate results (26) and (28) are acceptable only in the limits of validity of assumptions (1)–(3) and of relation (24).

IV. CONCLUSIONS

We make the following observations:

(i) As discussed in the initial part of this article, the effect of radiant pressure and the thermal effect (the "kicking off" of the molecules from the two surfaces) tend to push the radiometer leaves in opposite directions. In the derivation of (26) from (23) one finds that the radiant pressure effect is of order $(\beta^{1/2}c)^{-1}$ with respect to the thermal effect. Since $(\beta^{1/2}c)^{-1}$ equals the ratio of the average thermal speed to the speed of light, one sees that the effect of radiant pressure is indeed negligible.

(ii) For a given radiation flow J_e, the speed of motion of the leaf is inversely proportional to the molecular concentration n of the background gas. This is one of the reasons for operating a radiometer in rarefied air.

(iii) The speed of the leaf is inversely proportional to the temperature of the gas and is independent of the mass m of the gas molecules. Then, for a rarefied gas, the speed of the leaf is inversely proportional to the pressure.

(iv) Somewhat surprisingly, we find that the motion of the leaf is not strongly affected by the value of the accommodation coefficients.[6]

(v) Relation (25b) shows the "heating up" of the front edge. Relation (25a) shows the competing effects of molecular interaction and photon energy flow upon the temperature of the trailing surface. Both effects were described under point (A) of Sec. II.

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APPENDIX

For K a nonnegative integer,[7] let

$$I_K = \int_0^\infty dx\, x^K \exp(-x^2) = \tfrac{1}{2}\Gamma\left(\frac{K+1}{2}\right).$$

Observe that

$$I_0 = \int_0^\infty dx \exp(-x^2) = \frac{\pi^{1/2}}{2}$$

and that

$$I_K = [(K-1)/2]I_{K-2}.$$

Then, by induction,

$$I_{2n} = [(2n-1)!!/2^n](\pi^{1/2}/2),$$

where

$$(2n-1)!! = (2n-1)(2n-3)\cdots 1.$$

Also, from the properties of the gamma function,[7]

$$I_{2K+1} = \tfrac{1}{2}\Gamma(K+1) = K!/2.$$

[1]Available from Windsor Electronics, Inc., PO Box 662, Wheaton, IL 60187. The device is also known in the literature as the "vane radiometer" and as "William Crooks's radiometer."

[2]Thanks are due to Professor Kuščer for suggesting the most important references.

[3]A review of the work done in the earlier part of this century on "radiometric effects" may be found in L. B. Loeb, *The Kinetic Theory of Gases* (McGraw-Hill, New York, 1934; Dover, New York, 1961), Sec. 84; M. Knudsen, *Kinetic Theory of Gases* (Methuen, London, 1934; Wiley, New York, 1934; reprinted 1952). Knudsen's "absolute manometer," which operates on the basis of radiometric effects, is described by R. Jaeckel, in *Encyclopedia of Physics*, edited by S. Flugge (Springer, Berlin, 1958), Vol. XII, pp. 530–3.

[4]Maxwell's accommodation model for gas–surface interactions is described, for instance, in J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972), Sec. 15.1. More sophisticated models are a subject of current research. For example, see I. Kuščer and H. Lang, Z. Naturforsch. A **28**, 1468 (1973); I. Kuščer, Conf. Sem. Mat. Università Bari **135**, 1 (1974), and references quoted therein.

[5]Radiometric effects for high-pressure conditions have been studied by several authors: see Loeb (Ref. 3) and Jaeckel (Ref. 3), and references quoted therein.

[6]One can show, however, that the leaf speed decreases significantly for $\alpha_1 + \alpha_2 \simeq 0$ if assumption (1) is relaxed—that is, if heat conduction across the leaf is included in the computation.

[7]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Natl. Bur. Stand., Washington, DC, 1964), Sec. 6.

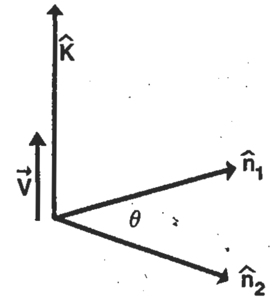

Fig. 1. Unit vectors $\hat{n}_1$ and $\hat{n}_2$ lying in the xy plane and directed along two axes intersecting at an angle θ. The rotation vector **V** corresponding to a π rotation about the $\hat{n}_1$ direction followed by a π rotation about the $\hat{n}_2$ direction is shown lying along the z axis.

while the square dot product is commutative,

$$\mathbf{V}_2 \boxdot \mathbf{V}_1 = \mathbf{V}_1 \boxdot \mathbf{V}_2. \tag{13}$$

For two equal parallel rotations we have

$$\mathbf{V} \boxtimes \mathbf{V} = 2f\mathbf{V}. \tag{14}$$

For a rotation and its reciprocal we have **V** and −**V**, respectively:

$$\mathbf{V}^{-1} = -\mathbf{V}, \tag{15}$$

which gives

$$\mathbf{V} \boxtimes (-\mathbf{V}) = 0. \tag{16}$$

No rotation leaves **V** unchanged:

$$\mathbf{V} \boxtimes 0 = \mathbf{V}, \quad \mathbf{V} \boxdot 0 = f. \tag{17}$$

As an example of the advantage of this product, let us solve the problem presented in the introduction, namely, that of two rotations through an angle π about two axes intersecting at an angle θ. The first π rotation about the x axis, for example, is represented by

$$\mathbf{V}_1 = \hat{n}_1 = \hat{i}, \quad f_1 = 0.$$

It is followed by a π rotation about an axis in the xy plane oriented at an angle θ relative to the x axis which is represented by

$$\mathbf{V}_2 = \hat{n}_2 = \hat{i}\cos\theta + \hat{j}\sin\theta, \quad f_2 = 0.$$

These axes are shown in Fig. 2. The composite rotation is given by Eq. (10):

$$\mathbf{V} = \mathbf{V}_2 \boxtimes \mathbf{V}_1 = \mathbf{V}_1 \times \mathbf{V}_2 = \hat{k}\sin\theta = \hat{k}\sin(2\theta/2).$$

Thus the equivalent single rotation is through an angle 2θ about the $\hat{k}$ direction. The vector **V** is shown in Fig. 1.

THREE SUCCESSIVE ROTATIONS

The result of Eq. (10) can easily be expanded to three or more successive rotations. In particular, it is useful to have an expression for three successive rotations,

$$\begin{aligned}\mathbf{V}_3 \boxtimes \mathbf{V}_2 \boxtimes \mathbf{V}_1 = {} & f_1(\mathbf{V}_2 \times \mathbf{V}_3) + f_2(\mathbf{V}_1 \times \mathbf{V}_3) \\ & + f_3(\mathbf{V}_1 \times \mathbf{V}_2) - \mathbf{V}_1(\mathbf{V}_2 \cdot \mathbf{V}_3) + \mathbf{V}_2(\mathbf{V}_1 \cdot \mathbf{V}_3) \\ & - \mathbf{V}_3(\mathbf{V}_1 \cdot \mathbf{V}_2) + f_2 f_3 \mathbf{V}_1 + f_1 f_3 \mathbf{V}_2 + f_1 f_2 \mathbf{V}_3.\end{aligned} \tag{18}$$

As we can see, Eq. (18) shows that the triple product is associative.

ROTATION OF A VECTOR

So far we have shown the resultant of two successive rotations; now we will see how we rotate a vector. The vector denoted by **U** has a magnitude U and a direction $\hat{n}$:

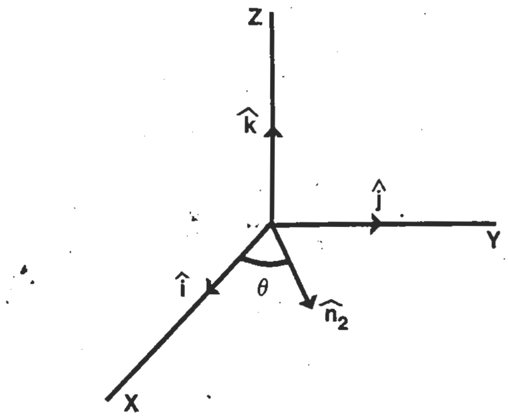

Fig. 2. Unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ along the three Cartesian coordinate directions and unit vector $\hat{n}_2$ lying in the xy plane.

$$\mathbf{U} = U\hat{n}. \tag{19}$$

We will carry out the rotation on the unit vector $\hat{n}$ to produce a new unit vector $\hat{n}'$ oriented in a different direction. Since the magnitude U is unaffected by the rotation, we obtain

$$\mathbf{U}' = U\hat{n}'.$$

The unit vector $\hat{n}$ is in the form of a rotation vector, and so it can be transformed by Eq. (18) with $\mathbf{V}_1$ equal to the reciprocal of $\mathbf{V}_3$:

$$\hat{n}' = \mathbf{V} \boxtimes \hat{n} \boxtimes \mathbf{V}^{-1} = \mathbf{V} \boxtimes \hat{n} \boxtimes (-\mathbf{V}), \tag{20}$$

where we used the property that $\mathbf{V}^{-1} = -\mathbf{V}$ from Eq. (15). Making use of Eq. (18), we obtain

$$\hat{n}' = 2f\hat{n} \times \mathbf{V} + 2\mathbf{V}(\mathbf{V} \cdot \hat{n}) + \hat{n}(1 - 2V^2). \tag{21}$$

As another example of the efficiency of this type of vector product, we will use Eq. (21) to solve the problem discussed by Palazzolo[3] of rotating a coordinate system around the [1 1 1] axis by the angles $\phi = 120°$ and $\phi \neq 120°$, as shown in Fig. 3.

The rotation vector **V** in the [1 1 1] direction is given by

$$\mathbf{V} = [(\hat{i} + \hat{j} + \hat{k})/3^{1/2}]\sin(\phi/2), \tag{22}$$

where ϕ is the rotation angle ($\phi = 120°$ for the Marion[4] example). To find the effect of this rotation on the coordinate system, we need to apply it to each one of the axes. We start with the x axis. The unit vector $\hat{n}$ along the x axis is given by

$$\hat{n} = \hat{i}. \tag{23}$$

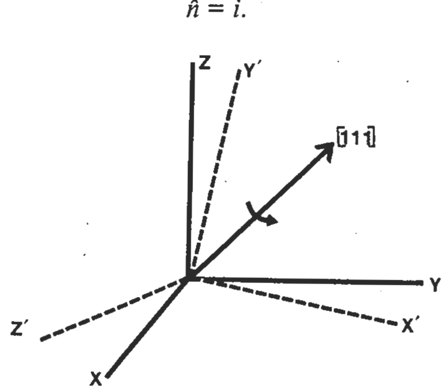

Fig. 3. Rotation about the [1 1 1] direction through an arbitrary angle. The x,y,z axes before and the x',y',z' axes after the rotation are shown.

Applying Eq. (21), we obtain

$$\hat{n}' = (\hat{i}/3)(1 + 2\cos\phi) + (\hat{j}/3)(1 - \cos\phi - 3^{1/2}\sin\phi) + (\hat{k}/3)(1 - \cos\phi + 3^{1/2}\sin\phi). \quad (24)$$

For the case where $\phi = 120°$ we have

$$\hat{n}' = \hat{k}, \quad (25)$$

and for the case where $\phi = 119°$

$$\sin(119°) = 0.8746,$$
$$\cos(119°) = 0.4848,$$
$$\hat{n}' = 0.01013\hat{i} - 0.01002\hat{j} + 0.99988\hat{k}. \quad (26)$$

Of course, for the y and z axes the procedure is analogous. With slightly more complication the problem of rotating not only around a [1 1 1] direction but around any direction can be easily solved with the help of Eq. (21).

CONTINUOUS ROTATIONS

The previous sections described successive individual rotations. Another important case involves continuous rotations where the rotation vector is a function of time.

To illustrate the continuous rotation case we will consider the time evolution of a vector $\mathbf{V}_{(t)}$ which represents the accumulative rotation from a time $t = 0$ to a time t. A short interval of time Δt later the vector $\mathbf{V}_{(t+\Delta t)}$ will be related to $\mathbf{V}_{(t)}$ by an incremental rotation vector $\mathbf{V}_{(\Delta t)}$ which rotates through an angle $\Delta\phi$ in the time Δt about an instantaneous direction defined by the unit vector $\hat{n}$. Therefore, from Eq. (7) we may write

$$\mathbf{V}_{(t+\Delta t)} = \mathbf{V}_{(\Delta t)} \boxtimes \mathbf{V}_{(t)}. \quad (27)$$

The rotation vector $\mathbf{V}_{(t)}$ has the associated scalar $f(t)$.

The incremental rotation vector $\mathbf{V}_{(\Delta t)}$ is a function of t and Δt and has the form

$$\mathbf{V}_{(\Delta t)} = \hat{n}\sin(\Delta\phi/2) \approx (\hat{n}\Delta\phi/2) \approx (\hat{n}\omega\Delta t/2) \approx (\omega\Delta t/2), \quad (28)$$

where we assume an incremental rotation ($\Delta\phi \ll 1$) with the angular velocity ω during the time Δt, and ω is in the direction of $\hat{n}$. The factor $f_{(\Delta t)}$ associated with $\mathbf{V}_{(\Delta t)}$ is essentially unity:

$$f_{(\Delta t)} = (1 - V_{(\Delta t)}^2)^{1/2} \approx (1 - \omega^2\Delta t^2/4)^{1/2} \approx 1, \quad (29)$$

since $\omega\Delta t \ll 1$.

With the aid of Eqs. (28) and (29) we may use Eq. (10) to write Eq. (27) as

$$\mathbf{V}_{(t+\Delta t)} = (\Delta t/2)\mathbf{V}_{(t)} \times \omega + \mathbf{V}_{(t)} + (f\Delta t/2)\omega. \quad (30)$$

The time derivative of $\mathbf{V}$ is

$$\dot{\mathbf{V}} = (\mathbf{V}_{(t+\Delta t)} - \mathbf{V}_{(t)})/\Delta t,$$

which becomes, using Eq. (30),

$$\dot{\mathbf{V}} = (1/2)\mathbf{V} \times \omega + (f/2)\omega. \quad (31)$$

We can form the products $\mathbf{V} \cdot \dot{\mathbf{V}}$ and $\mathbf{V} \times \dot{\mathbf{V}}$,

$$\mathbf{V} \cdot \dot{\mathbf{V}} = (f/2)\mathbf{V} \cdot \omega \quad (32)$$

and

$$\mathbf{V} \times \dot{\mathbf{V}} = (1/2)\mathbf{V}(\mathbf{V} \cdot \omega) - (1/2)V^2\omega + (f/2)\mathbf{V} \times \omega, \quad (33)$$

and the factors $\mathbf{V} \times \omega$ from Eq. (31) and $\mathbf{V} \cdot \omega$ from Eq. (32) may be substituted into Eq. (33) to give

$$\mathbf{V} \times \dot{\mathbf{V}} = \mathbf{V}(\mathbf{V} \cdot \dot{\mathbf{V}})/f - (1/2)\omega(V^2 + f^2) + f\dot{\mathbf{V}}. \quad (34)$$

This may be simplified by observing that $V^2 + f^2 = 1$ from Eq. (4), and

$$\dot{f} = \frac{d}{dt}(1 - V^2)^{1/2} = -\frac{2}{f}\mathbf{V} \cdot \dot{\mathbf{V}} \quad (35)$$

to give, solving for ω in Eq. (34),

$$\omega = 2[\dot{\mathbf{V}} \times \mathbf{V} + f\dot{\mathbf{V}} - (\dot{f}/2)\mathbf{V}]. \quad (36)$$

Thus we have obtained an expression for the instantaneous rotation axis in terms of the time dependence of a rotation vector $\mathbf{V}$.

APPENDIX

The general rotation through an angle ϕ about an arbitrary axis whose direction is defined by the unit vector $\hat{n}$ may be written[5]

$$R = \exp[(\theta/2)i\hat{n} \cdot \sigma], \quad (A.1)$$

where σ is a vector operator whose components are the Pauli spin matrices,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (A.2)$$

The exponential of Eq. (A.1) may be expanded in a power series, and the terms grouped to give

$$R = \cos(\phi/2) + i\hat{n} \cdot \sigma \sin(\phi/2), \quad (A.3)$$

where we understand that a 2×2 unit matrix multiplies the $\cos(\phi/2)$ term.

In order to treat two successive rotations, we write their corresponding rotation operators R_1 and R_2 in the form of Eq. (A.3),

$$R_1 = \cos(\phi_1/2) + i\hat{n}_1 \cdot \sigma \sin(\phi_1/2)$$

and

$$R_2 = \cos(\phi_2/2) + i\hat{n}_2 \cdot \sigma \sin(\phi_2/2), \quad (A.4)$$

and multiply them together to give

$$R_3 = R_2R_1 = \cos(\phi_2/2)\cos(\phi_1/2) + i\hat{n}_1 \cdot \sigma \sin(\phi_1/2)\cos(\phi_2/2) + i\hat{n}_2 \cdot \sigma \sin(\phi_2/2)\cos(\phi_1/2) - (\hat{n}_1 \cdot \sigma)(\hat{n}_2 \cdot \sigma)\sin(\phi_1/2)\sin(\phi_2/2). \quad (A.5)$$

We can make use of the property

$$(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}) \cdot \sigma \quad (A.6)$$

to write Eq. (A.6) in the form

$$R_3 = [\cos(\phi_1/2)\cos(\phi_2/2) - \hat{n}_2 \cdot \hat{n}_1 \sin(\phi_1/2)\sin(\phi_2/2)] + i\hat{n}_1 \cdot \sigma \sin(\phi_1/2)\cos(\phi_2/2) + i\hat{n}_2 \cdot \sigma \sin(\phi_2/2)\cos(\phi_1/2) - (\hat{n}_2 \times \hat{n}_1) \cdot \sigma \sin(\phi_1/2)\sin(\phi_2/2). \quad (A.7)$$

Since R_3 is a rotation operator, it can be written in the form of Eq. (A.3):

$$R_3 = \cos(\phi_3/2) + i\hat{n}_3 \cdot \sigma \sin(\phi_3/2). \quad (A.8)$$

A comparison of Eqs. (A.7) and (A.8) leads to the identification

$$\cos(\phi_3/2) = \cos(\phi_1/2)\cos(\phi_2/2) - \hat{n}_1 \cdot \hat{n}_2 \sin(\phi_1/2)\sin(\phi_2/2); \quad (A.9)$$

$$\hat{n}_3 \sin(\phi_3/2) = \hat{n}_1 \sin(\phi_1/2)\cos(\phi_2/2) + \hat{n}_2 \sin(\phi_2/2)\cos(\phi_1/2) + \hat{n}_1 \times \hat{n}_2 \sin(\phi_1/2)\sin(\phi_2/2). \quad (A.10)$$

These expressions (A.9) and (A.10) were obtained earlier by Halpern[6] and Schwinger[7] using a different method. If we examine Eq. (A.10), we see that it can be put in the form

$$\mathbf{V}_3 = \mathbf{V}_1 \times \mathbf{V}_2 + \mathbf{V}_1 \cos(\phi_2/2) + \mathbf{V}_2 \cos(\phi_2/2), \quad (A.11)$$

and Eq. (A.9) becomes

$$f_3 = f_1 f_2 - \mathbf{V}_1 \cdot \mathbf{V}_2, \quad (A.12)$$

where

$$f_i = \cos(\phi_i/2). \quad (A.13)$$

Combining Eqs. (A.11) and (A.13), we have

$$\mathbf{V}_3 = \mathbf{V}_1 \times \mathbf{V}_2 + \mathbf{V}_1 f_2 + \mathbf{V}_2 f_1. \quad (A.14)$$

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