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INSTANTONS AND CONFINEMENT *

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The Euclidean vacuum functional of Yang-Mills theory is saturated by infinitely large two-dimensional sheets of calculable universal thickness. The gauge fields inside the sheets are characterised by a distribution of instantons held together by a smooth solution of the field equations. The sheets cannot grow in the transverse directions. Within each sheet the orientation in group space of the average field is correlated to the spatial orientation of the area, while different sheets are totally uncorrelated. We exhibit a class of fluctuations above this background which confine quarks. No assumption about the infra-red behaviour of the Gell-Mann–Low function is used; all quantities are calculable and no free parameters exist beyond the group structure constants.

1. Introduction

It has been speculated for some time that the infra-red structure of Yang-Mills theories is dominated by large-scale random fields. The arbitrariness of the orientation in group space would then destroy the long-range correlations characteristic of weakly coupled vector fields and provide a mechanism for quark confinement.

In the framework of lattice gauge theory [1,2] the above situation prevails automatically in the strong-coupling limit and the main problem is to show that the long-range correlations are not restored in the weak-coupling limit dictated by asymptotic freedom.

It has been conjectured by Polyakov [3] that the existence of classical Euclidean solutions of finite action in Yang-Mills theory [3,4] may provide a mechanism for the destruction of the long-range correlations in the continuum quantum field theory **. This conjecture has been demonstrated for certain three-dimensional (2 + 1) models [6] whose infra-red behaviour is essentially that of an Abelian vector field interacting with magnetic monopoles. The vacuum in this case is dominated by the plasmons of the three-dimensional monopole Coulomb gas.

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** An approach which does not use exact classical solutions was proposed in ref. [5].

We propose, in what follows, to show that the multi-instanton fields do indeed generate a large class of configurations, stable under a wide range of variations whose weight in the Euclidean vacuum functional exceeds that of the perturbative vacuum fluctuations. If these configurations are assumed to saturate the vacuum (in the sense of the standard variational approach), then the latter supports a class of quantum fluctuations which confine quarks.

More specifically, it will be shown (sect. 2) that the contribution of a multi-instanton configuration to the functional integral possesses a maximum at a particular value of the local instanton density and the global size and shape. We thus approximate the vacuum functional by a "gas" whose "molecules" are lumps of instantons *or* anti-instantons. In other words, we suggest that instantons (or anti-instantons) bind together to form "super-molecules" which are approximately non-interacting and fill up space time. The size of the lumps provides a macroscopic scale while the inter-instanton distance inside them fixes the effective coupling of infra-red fluctuations. It is found that the relevant local parameters (density, instanton radius and effective coupling) are still small so that the semi-classical approximation is self consistent.

In sect. 3 we identify a class of fluctuations which are capable of forcing the averaged Wilson-loop to obey an area law. These are infinitely long randomly distributed quantized fluxons (which form two-dimensional surfaces in Euclidean space-time). The quantization is due to the requirement that the long range $1/r$ vector potential outside the fluxon leave the non-Abelian instanton fields univalued (this is the same mechanism which quantizes flux in a superconductor; in our case the non-Abelian background plays the role of the Higgs field). The background fields provide the necessary scale for the fluxons, and a crude estimate leads to the conclusion that they are indeed open-ended (and infinite). This in turn implies the area law, while the fluxon density determines the slope of the linear confining potential between triality carrying external sources.

In sect. 4 we briefly discuss the validity of the saturation assumption and indicate directions for extension and improvement. A short appendix contains the derivation of an exact, convenient, general representation of functional integrals by an integral over collective coordinates.

We emphasize that no assumption is made about the infra-red behaviour of the Gell-Mann–Low function, which is used only in the ultra-violet domain where asymptotic freedom holds true. Throughout the discussion, we assume that the gauge group is SU(2) and no attempt is made to include SU(3) effects.

2. The Yang-Mills vacuum

The SU(2) pure Yang-Mills theory is specified by a massless vector potential $A^a_\mu(x)$ $(a = 1, 2, 3)$, its field strength,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \tag{1}$$

and the classical Euclidean action functional,

$$\mathcal{E}(A) = \tfrac{1}{4}\int d^4x (F^a_{\mu\nu})^2 \, . \tag{2}$$

The objects of interest in the quantised theory are the gauge invariant Euclidean functional integrals:

$$\langle f(A)\rangle = [\int dA \, e^{-g_0^{-2}\mathcal{E}(A)} f(A)] \, [\int dA \, e^{-g_0^{-2}\mathcal{E}(A)}]^{-1} \, , \tag{3}$$

where g_0 is the bare cut-off dependent dimensionless coupling constant. As is well-known [7], due to the gauge invariance of the action, it is necessary to reduce the number of independent integration variables in eq. (3) by a gauge-fixing procedure. This has the effect of replacing eq. (3) by

$$dA \to dA \, \Delta(A) \, \delta[G^a(A)] \, , \tag{4}$$

where $G^a(A)$ are three arbitrary local functions of A, while the Fadeev-Popov determinant Δ is defined by

$$\Delta(A) = \det\left[\frac{\delta G^a}{\delta A^c_\mu} D^{cb}_\mu\right] \, , \tag{5}$$

and the covariant derivative D_μ is,

$$D^{ab}_\mu(A) = \delta^{ab}\partial_\mu + \epsilon^{acb} A^c_\mu \, . \tag{6}$$

Yang-Mills theory is asymptotically free [8] which means that $g_0 \to 0$ as the ultra-violet cut-off of the theory is removed. More specifically, if λ is an ultra-violet length scale, the renormalised coupling constant g_λ is defined by [8],

$$8\pi^2 g_\lambda^{-2} \equiv \epsilon(\lambda) = \tfrac{22}{3} \ln \frac{\lambda_0}{\lambda} >> 1 \, , \tag{7}$$

where λ_0 is the physical unit of length. All the dependence on the cut-off is then removed by rewriting the action:

$$g_0^{-2}\,\mathcal{E}(A) \to g_\lambda^{-2}\,\mathcal{E}(A) + \text{counter terms} \, . \tag{8}$$

In particular, the integration over fields whose Fourier decomposition contains only momenta smaller than λ^{-1} is affected by simply dropping the counter terms, and cutting off the rest of the momenta at $p \sim \lambda^{-1}$.

Our aim is to identify a class of configurations $a^a_\mu(x)$ whose contribution dominates the functional integral (3). Consider a configuration $a^a_\mu(x, \zeta)$ which depends upon the parameters $\{\zeta_j\}$. The functional integral $dA^a_\mu(x)$ may be transformed into an integral over the parameters $\{\zeta\}$ and over modes orthogonal to their variation by a generalised Fadeev-Popov procedure (appendix A). For this purpose, we introduce the notations:

$$(f|g) = \int d^4x\, f(x)\, g(x)\,, \tag{9}$$

$$a_j = \frac{\partial a}{\partial \zeta_j}, \qquad a_{jk} = \frac{\partial^2 a}{\partial \zeta_j \partial \zeta_k}\,, \tag{10}$$

$$W_{ij} = (a_j | a_k)\,. \tag{11}$$

We then derive the exact formula (eq. (9)):

$$\int dA\, e^{-g^{-2}\mathcal{E}(A)} = \int d\zeta (\det W)^{1/2} \int d\eta\, \det[\delta_{ij} - W^{-1}_{ik}(a_{kj}|\eta)]$$
$$\times \Delta(a+\eta)\, e^{-g^{-2}\mathcal{E}(a+\eta)}\,, \tag{12}$$

where the modes $\eta^a_\mu(x)$ satisfy

$$(a_j|\eta) = 0\,. \tag{13}$$

Our objective will thus be achieved if we identify a set of parameters and configurations which maximise W (entropy) at a minimal cost in $\mathcal{E}$ (energy). The latter objective is clearly met by looking for solutions of the field equations. We shall approach the maximisation of W by choosing for the parameters $\{\zeta\}$ a set of points $\{z_j\}$ in coordinate space at which the action density achieves a local maximum, thus generating large derivatives. If the average distance, l, between neighbouring maxima is large compared to their width α, the off-diagonal terms of W_{ij} (eq. (11)) and the second derivatives in $(a_{kj}|\eta)$ may be neglected, leading to

$$(\det W)^{1/2} \simeq \prod_j w(z_j)\,. \tag{14}$$

We shall further require (and verify *a posteriori*) that the length scales α and l be ultra-violet, namely that both $\epsilon(\alpha)$ and $\epsilon(l)$ (eq. (7)) be large compared to unity. Finally, we shall neglect the variation of all the orthogonal modes η whose wavelengths exceed the inter-maxima distance l. These properties allow the integral $(d\eta)$ to be performed by renormalised perturbation theory.

As is well-known, the weight of the chosen configuration should be compared to that of classical vacuum ($A_\mu = 0$) fluctuations on the same scale. The above discussion therefore leads to the following representations of the functional integral (3) [6]:

$$\frac{[\int \mathrm{d}A\, \mathrm{e}^{-g_0^{-2}\mathcal{E}(A)}]_{A\sim a}}{[\int \mathrm{d}A\, \mathrm{e}^{-g_0^{-2}\mathcal{E}(A)}]_{A\sim 0}} \equiv \mathrm{e}^{-\mathcal{F}(a)} \simeq \frac{1}{N!}\int \prod_{j=1}^{N} \mathrm{d}z_j w_j\, \mathrm{e}^{-g_j^{-2}\mathcal{E}_j} . \tag{15}$$

g_j^{-2} is the inverse coupling (eq. (7)) at the scale α_j while $\mathcal{E}_j$ is the contribution of the local maxima at z_j to $\mathcal{E}$:

$$\mathcal{E}_j = \tfrac{1}{4} \int_{|x-z_j|\lesssim \frac{1}{2}l_j} \mathrm{d}^4x (f^a_{\mu\nu})^2 . \tag{16}$$

The relative weight w_j is computed by solving the small oscillation problem in the vicinity of the point z_j:

$$w_j \simeq \Delta(a_j)\left|\int_{|x-z_j|\lesssim \frac{1}{2}l_j} \mathrm{d}^4x\, \partial_\mu a\, \partial_\nu a\right| \left|\frac{M(0)}{M(a)}\right|^{1/2} . \tag{17}$$

M is the determinant of the second variation of the action around the indicated classical field computed under the assumption that the fluctuating modes η vanish at a distance $\frac{1}{2}l_j$ from the point z_j. We remark in this connection, that no zero modes exist in our problem since the translational modes have already been separated (eq. (17)), while the local scales α_j will be fixed by a variational calculation; moreover, our configurations will be seen to fix the relative orientation in group space of the maxima.

Having described our procedure, let us now introduce the proposed vacuum-saturating configurations. Consider the N-instanton singular field introduced by 't Hooft [9]

$$a^a_\mu = -\eta^a_{\mu\nu}\, \partial_\nu \ln \varphi , \tag{18}$$

where $(\bar\eta)\eta$ is the (anti)instanton symbol used by BPST [4]:

$$\eta^a_{4k} = \delta_{ak} = -\eta^a_{k4} = -\bar\eta^a_{4k} , \qquad \eta^a_{kl} = \epsilon_{akl} = \bar\eta^a_{kl} , \tag{19}$$

and $\varphi(x)$ is

$$\varphi(x) = L^{-2} + \sum_{j=1}^{N} \frac{\mu_j^2}{(x-z_j)^2} , \tag{20}$$

where L is a length scale (which will shortly be eliminated) and μ_j^2 are dimensionless parameters normalised by

$$\max \mu_j^2 = 1 \ . \tag{21}$$

In order to identify the field at $x \sim z_j$, the function $\varphi(x \sim z_j)$ will now be split into two parts:

$$x \sim z_j: \qquad \varphi(x) \simeq \frac{\mu_j^2}{(x-z_j)^2} + \alpha_j^{-2}\mu_j^2 \ , \tag{22}$$

where the length scale α_j is defined through

$$\alpha_j^{-2} = \Phi(z_j) + L^{-2} \ , \tag{23}$$

and the "potential" $\Phi(z)$ is defined by

$$\Phi(z) = \sum_{z_l \neq z} \frac{\mu_l^2}{(z-z_l)^2} \ . \tag{24}$$

In eqs. (22)–(24) it was assumed that $|x - z_j| << l_j$, the distance to the next singularity, so that the derivatives of Φ may be neglected compared to those of $(x-z)^{-2}$. The vector potential due to (22) is that of a singular (anti)instanton with radius α_j:

$$x \sim z_j: \qquad a_\mu^a(x) \simeq 2\eta_{\mu\nu}^a \frac{(x-z_j)_\nu}{(x-z_j)^2\,[1+\alpha_j^{-2}(x-z_j)^2]} , \tag{25}$$

while the action density is given by

$$x \sim z_j: \qquad \tfrac{1}{4}(f_{\mu\nu}^a)^2 = 48\,\frac{\alpha_j^{-4}}{[1+\alpha_j^2(x-z_j)^2]^4} \ . \tag{26}$$

Before continuing with the main argument, we remark that the vector potential a_μ^a (eq. (25)) is singular and therefore unacceptable as a contributor to the functional integral (ultra-violet fluctuations of scale 0 have negligible weight due to $g_{\lambda=0}^{-2} \to \infty$). However, it is possible [10,11] to construct from the field a_μ a regular solution $\tilde{a}_\mu$ of the field equations which is gauge equivalent to the singular field at all non-singular points $x \neq z_j$. This is achieved by defining a global gauge transformation which, in the vicinity of each z_j, turns the singular instanton (25) into a regular anti-instanton:

$$\tilde{a}_\mu^a \tau^a = g^{-1}(x)\,[a_\mu^a \tau^a + i\partial_\mu]\,g(x) \ , \tag{27}$$

where $g(x)$ satisfies:

$$x \sim z_j: \qquad g(x) \simeq |x - z_j|^{-1}((x - z_j)_4 - i(x - z_j)_a \tau^a) , \tag{28}$$

and

$$\tilde{a}^a_\mu \simeq 2\bar{\eta}^a_{\mu\nu} \frac{(x - z_j)_\nu}{[\alpha_j^2 + (x - z_j)^2]} . \tag{29}$$

Since gauge invariant quantities are unaffected by (27), (28) outside the singular points, the quantities $\mathcal{E}_j$ and w_j * (eqs. (16) and (17)) may be calculated by using (25), (26) and we shall not require any information about $g(x)$ beyond its existence.

Eq. (29) shows that provided $\alpha_j << l_j$ we shall have **

$$\mathcal{E}_j \simeq 8\pi^2 . \tag{30}$$

Moreover, since the local fields are those of a single instanton, the weight w_j may be approximated by that of a free instanton of radius α_j (as explained above, we should actually use for this calculation the approximation of enclosing the local instanton in a box of radius $\frac{1}{2}l$). Using the value of $w(\alpha)$ derived by 't Hooft [12], we obtain for the instanton density at the vicinity of z,

$$\rho(z) = \frac{2}{\pi^2 \alpha^4(z)} k\, \epsilon^4(\alpha(z))\, \mathrm{e}^{-\epsilon(\alpha(z))} . \tag{31}$$

The inverse coupling constant $\epsilon(\alpha)$ represents the one pertaining to renormalised dimensionally regularised perturbation theory and the numerical value of k is

$$k \simeq \tfrac{1}{3} \cdot \tfrac{3}{10} \cdot 4\mathrm{e}^7 \simeq 400 . \tag{32}$$

The extra factor $\frac{3}{10}$ (compared to eq. (13.8) of ref. [12]) arises from having integrated over all scales smaller than α (using eq. (7)) while the $\frac{1}{3}$ is our guess for the reduction due to the loss of the invariance under global SU(2) rotations. We now note that the density (31) increases with α so that configurations with larger α's dominate. Thus, the parameters L^2 and $\{\mu^2\}$ in eq. (23) should be chosen as large as possible, namely,

$$L^2 \to \infty , \qquad \mu_j^2 = 1 . \tag{33}$$

(Strictly speaking, we should integrate over L^2 and μ and include the appropriate determinants. However, we have already integrated over scales in deriving eq. (32) so that doing it again would be double-counting.)

* The gauge invariance of w_j is manifest in the expressions given in ref. [6].

** Actually $\mathcal{E} = N\, 8\pi^2$ exactly (ref. [9]).

We now come to the central physical effect of our configurations. We shall show that the local ultra-violet scales α_j are completely determined by the global properties of the singularity distribution, and *vice-versa*. Furthermore, it will be seen that the minimisation of the free energy $\mathcal{F}$ (defined by eq. (15)) leads to a complete determination of all the parameters (in particular, $\epsilon(\alpha)$). For this purpose, let us represent the density of singularities $\rho(z)$ by a continuous function and approximate the sum defining Φ (eq. (24) with $\mu = 1$) by an integral

$$\Phi(z) = \int_V \mathrm{d}^4 y \, \frac{\rho(y)}{(x-y)^2} \,, \tag{34}$$

where V is the (four-dimensional) volume occupied by the singularities. Since the integral (34) is convergent, the approximation is legitimate as long as $|z - z_j| \gg \alpha_j$. We remark that eq. (34) is the four-dimensional Coulomb potential due to the positive charge density $\rho(z)$:

$$\Box \Phi = 4\pi^2 \rho \,, \qquad \int_V \mathrm{d}z \, \rho(z) = N \,. \tag{35}$$

Using $L^{-2} \to 0$ and $\mu = 1$, we have for the local instanton radius

$$\alpha^{-2}(z) = \Phi(z) \,. \tag{36}$$

Eqs. (7), (31), (34), (36) constitute a coupled set of equations which determine the average scale α as a function of the shape and size of the domain V. In fact, suppose we approximate $\rho(z)$ by a constant, and designate the largest linear dimension of V by R. We then find

$$\alpha^{-2} = \Phi \simeq \frac{N}{R^2} = \rho(\alpha) \frac{V}{R^2} \,, \tag{37}$$

$$\rho(\alpha) = \frac{2}{\pi^2 \alpha_0^4} 400 \, \epsilon_\alpha^4 \, \mathrm{e}^{-5\epsilon_\alpha/11} \,. \tag{38}$$

Clearly $\rho(\alpha)$ decreases quickly to zero when $\alpha \to 0$. Therefore, if $V/R^2 \to \infty$ when $R \to \infty$ the density goes down, and the weight of the configuration decreases correspondingly. We conclude that it is impossible to fill up space-time with the instantons which reside in our ansatz. Rather, the optimal domain which is occupied by our configuration is an infinite two-dimensional sheet of finite (2-dimensional) cross section Λ. In order to determine the latter, we substitute for V a disc of radius R and thickness Λ to obtain

$$\frac{\alpha^2}{\Lambda^2} = \pi^2 \rho(\alpha) \, \alpha^4 \,. \tag{39}$$

Actually, eqs. (37), (39) are imprecise for a two-dimensional sheet since the Coulomb potential of such a distribution contains a logarithmic factor. In fact, the potential $\Phi(z)$ at the centre of our disc is

$$\frac{R}{\Lambda} \gg 1: \qquad \Phi(0) = \frac{N}{R^2}\left[1 + \ln\frac{R^2}{\Lambda^2}\right] . \tag{40}$$

However, eq. (39) shows that the logarithmic factor becomes significant only at distances of order $R \sim \Lambda \exp \sigma^{-1}$. As will be seen, the relevant values of σ^{-1} are very large so that henceforth we shall disregard all logarithmic considerations. It still remains for us to determine the optimum value for the instanton scale α which, as we have shown, fixes both the density and the thickness of the sheet. Note first that the free energy $\mathcal{F}$, eq. (15), corresponding to the density ρ is simply equal to N so the free energy density is equal to ρ:

$$-f(z) = \rho(\alpha(z)) . \tag{41}$$

Up to now, we have disregarded the fields outside the volume V. It is easy to verify that for a single sheet the total contribution to the classical action from the fields outside the boundaries is $\sim 8\pi^2$, namely negligible compared to $8\pi^2 N$. Therefore, we wish to consider the interaction between a large number of sheets embedded in space-time. The ansatz we shall choose for the combined vector potential will be quite simple. In order to fill up space as much as possible with the free energy density (41), the vector potential will be forced by hand to vanish within a distance $\frac{1}{2}l$ from the surface of the sheet. (The choice $\frac{1}{2}l$ is dictated by the requirements that the surface fields be as small as possible and that a minimal fraction of space-time be empty of instantons.) In order to achieve this, the local field near each surface instanton will be brought to zero within a shell of radius $\sim\frac{1}{4}l$ and thickness $\sim\alpha$ around its centre. The estimated additional action per instanton:

$$g_\alpha^{-2} \Delta \mathcal{E}_{\text{sur.inst.}} \simeq 96\epsilon(\alpha)(\alpha/l)^3 . \tag{42}$$

The (three-dimensional) surface to volume ratio yields the number of surface instantons:

$$N_{\text{sur.}} = N\, 2l/\Lambda . \tag{43}$$

Substituting eq. (39) for Λ and using the geometrical relation $l = \alpha\rho^{-1/4}$ we find for the total free-energy density of sheet:

$$-f_{\text{total}} = \rho(\alpha)[1 - 192\pi\epsilon(\alpha)\,\alpha^4\rho(\alpha)] . \tag{44}$$

By using eqs. (7), (31), (32), it is readily seen that $f(\alpha)$ increases from zero as α is varied from zero, and reaches a maximum at a scale α_*,

$$\epsilon(\alpha_*) = 28.9 . \tag{45}$$

The physical parameters of the configurations are given by:

$$\alpha_*^4 \rho_* = 1.6 \times 10^{-5} \,, \tag{46}$$

$$l_* = 16\alpha_* \,, \tag{47}$$

$$\Lambda_* = 80\alpha_* \,. \tag{48}$$

Note that, according to eq. (7), if we use $\frac{1}{2}(l_* - 4\alpha_*) = 6\alpha_*$ as a representative for a typical inter-instanton scale, we find

$$\epsilon(6\alpha_*) = 15.7 >> 1 \,. \tag{49}$$

We conclude from eqs. (45)–(49) that the critical coupling ϵ_* and the scales (α_*, l_*) associated with it are indeed ultra-violet so that the perturbative treatment of the quantum fluctuations on this scale is justified. Moreover, the values (47), (48) for (l, Λ) are sufficiently large to justify the continuous approximation; in fact, the number of instantons within a sphere radius Λ is $N_\Lambda \sim 3200$ so that $\sqrt{N_\Lambda} = 57 << N_\Lambda$. Note also that the relative volume excluded by the instanton is $\sim 10^{-4}$ so that the neglect of the overlap is calculating the weight factor w is also safe. Further discussions of the stability of the sheets under quantum fluctuations will be given in sect. 4. We have thus reached our main result.

The Yang-Mills Euclidean vacuum functional is dominated by an essentially (see below) close-packed distribution of infinite (however, remember the log!) two-dimensional sheets of cross section Λ_* (eq. (48)) filled with a density ρ_* (eq. (64)) of correlated (anti)instantons. The gauge fields of different sheets are totally uncorrelated and their relative global orientation in group space is an undetermined collective coordinate which is to be integrated upon.

Up until now, we have assumed the sheets to be rigid. When performing the functional integration we should, however, take into account the harmonic fluctuations in the shape of the sheets. In order to estimate the action variation associated with these oscillations (which are obviously characterised by wavelengths larger than Λ_*) let us divide the total action into two parts:

$$\mathcal{E} = \sum_j \mathcal{E}_j + \mathcal{E}(\Phi) \,. \tag{50}$$

We shall now estimate $\mathcal{E}(\Phi)$ and show that it is indeed small compared to $\Sigma\, \mathcal{E}_j$. This we do by computing $(\Phi, a_\mu, f_{\mu\nu})$ at the centre of a ball of radius Λ. We readily find

$$\Phi(z) = \frac{N_{\Lambda *}}{\Lambda_*^2}\left(2 - \frac{z^2}{\Lambda_*^2}\right) , \tag{51}$$

$$z \sim 0: \qquad a_\mu^a(z) \simeq \frac{\eta_{\mu\nu}^a z_\nu}{\Lambda_*^2} \simeq -\eta_{\mu\nu}^a \partial_\nu \ln \Phi \,, \tag{52}$$

$$f^a_{\mu\nu}(0) = -\frac{2\eta^a_{\mu\nu}}{\Lambda_*^2}\,. \tag{53}$$

Thus, the density of the glue action is

$$\tfrac{1}{4}(f^a_{\mu\nu})^2 \simeq 12\Lambda_*^{-4}\,. \tag{54}$$

$\mathcal{E}(\Phi)$, therefore, is indeed small compared to $N\,8\pi^2$.

We conclude this discussion of the Yang-Mills vacuum by remarking that our calculations are clearly only a crude approximation even within our approach. A more complete treatment should use the free-energy density functional $f(\Phi)$ as a basis for a variational principle. Improvements are also desirable regarding the calculation of the deviation of the weight factor w from its dilute-gas value.

3. Confinement

As is well-known, the effective (Euclidean) action functional S_{I} which generates the interaction between quark–anti-quark pairs which are produced at a point, travel along a specified classical path and reannihilate, is given [1,6] by the vacuum expectation value of the ordered exponential of the line integral of the vector potential evaluated along the closed space-time loop:

$$e^{-S_{\mathrm{I}}[r(t)]} = \mathrm{tr}\langle(\exp i \oint_{\mathrm{L}} \mathrm{d}x_\mu A_\mu)\rangle \equiv \mathrm{tr}\langle U(\mathrm{L})\rangle\,, \tag{55}$$

where L is the loop generated by the closed orbit $r(t)$, while A_μ is the matrix $t^a A^a_\mu$ (t^a are representation matrices) and $(...)_+$ is the line ordering symbol.

Observe that since $U(\mathrm{L})$ is a bounded unitary matrix, S_{I} can increase as a function of loop size only if many approximately degenerate configurations contribute to the functional average in eq. (55). In such a case the various contributions may interfere destructively and reduce the matrix elements of $U(\mathrm{L})$. In view of eqs. (18)–(24) it is clear that local fluctuations in the instanton positions induce variations in the vector potential which decrease as R^{-3}, so that the phase of $U(\mathrm{L})$ will be sensitive only to nearby (distance $\sim\alpha$) instantons. The number of these is proportional to the circumference of the loop so that their effect is merely to renormalise the effective mass of the quarks [5]. We are thus led to look for a class of quantum fluctuations on top of the instanton background which, on the one hand, are local (in order to have a non-vanishing average density) and, on the other hand, induce long-range variations in the vector potential A_μ. To this end, consider a fluctuation which in the two-dimensional surface enclosed by the loop L is a pure gauge transformation everywhere except for a set of smeared singularities randomly distributed throughout the area.

It will now be shown that if the flux trapped at the singularities is properly quantised, on the one hand we shall have no interaction between the fluxons and the background field apart from the local interaction at the singular points. On the other hand, a non-trivial phase will be induced on a quark loop. In fact, choose a gauge transformation which, after traversing a loop around one of the singularities, returns to $\exp(2\pi i t)$ where t is a group generator. An example is **:

$$\Omega(x) = \prod_j \exp i\theta_j n_j^a t^a \,, \tag{56}$$

where n_j^a are arbitrary unit vectors (in three dimensions) and θ_j is the azimuthal angle at the point x with respect to the singularity at x_j;

$$(x - x_j) = |x - x_j|(\cos\theta_j, \sin\theta_j) \,. \tag{57}$$

The vector potential A_μ^a belongs to the adjoint representation of SU(2) so that a gauge transformation by (2π) does not affect it. If, however, the quark loop carries a representation with half-integer colour (in particular, if the quarks belong to the fundamental $\underline{2}$ representation) the matrix $U(\mathrm{L})$ will be non-trivial *:

$$U(\mathrm{L}) = (-)^{N_\mathrm{L}} \,, \tag{57}$$

where N_L is the total number of fluxons enclosed by the loop. Suppose now that the average density per unit area of the fluxons is μ^2 (μ is a mass) so that the distribution of N_L is Gaussian:

$$p(N_\mathrm{L}) \propto \exp\left(-\frac{(N_\mathrm{L} - \mu^2 \mathcal{A}_\mathrm{L})^2}{2\mu^2 \mathcal{A}_\mathrm{L}}\right) , \tag{58}$$

where $\mathcal{A}_\mathrm{L}$ is the area of the loop.

We then find (for $\mathcal{A}_\mathrm{L} \to \infty$):

$$\mathrm{tr}\langle U(\mathrm{L})\rangle = 2 \sum_{N_\mathrm{L}=0}^{\infty} p(N_\mathrm{L})\, \mathrm{e}^{i\pi N_\mathrm{L}} = 2\, \mathrm{e}^{-\pi^2\mu^2 \mathcal{A}_\mathrm{L}/2} \,. \tag{59}$$

Eq. (59) is the expression of quark confinement. The large distance force between a quark and an anti-quark is given by the variation of S_I with respect to the area and we therefore have an attractive force:

$$\text{force} = \tfrac{1}{2}\pi^2\mu^2 \,. \tag{60}$$

* The quantisation condition is similar to the one employed for merons [5]. The difference is that merons do produce long-range field strength and interact strongly with the background.

** These structures were also introduced by G. 't Hooft [13].

We shall now argue that the fluxon density is indeed non-zero and that its distribution on any given two-dimensional surface is random in the same sense of eq. (58). Note first that due to the conservation of flux (on a given Euclidean plane our fluxons are magnetic), the fluxons have to be closed. Therefore it would seem that they always appear in pairs and force N_L to be even (except for circumference contributions), thereby invalidating eq. (58). This argument, however, contains a loophole: the flux loops may become infinitely large thereby destroying the pairing correlations. The length of a typical fluxon is determined by the competition between the cost in action and the gain in entropy associate with the addition of an extra link to a pre-existing flux-line. A reliable calculation requires an ansatz for the fluxon carrying configurations supported by the instanton background. The integration over the extra collective coordinates would then be performed by using eq. (12). At this stage we are unable to carry out such a program and will thus resort to a very crude and hopefully reliable estimate.

We shall assume that the influence of the instanton background is summarized by two effects and neglect any further interaction between fluxons and instantons. First the radius of a typical fluxon will be taken to be of order $\sim\Lambda$. Second, since ultra-violet fluctuations are limited by the interinstanton distance l, the effective coupling constant will be fixed at the relevant value $g(l)$ (see eqs. (7) and (47)):

$$\frac{8\pi^2}{g^2(l)} \equiv \epsilon(l) \sim 10\ . \tag{61}$$

The average field-strength inside a fluxon is determined by the flux quantum:

$$F \sim \frac{2\pi}{\pi\Lambda^2}\,, \tag{62}$$

so that the action associated with a link of size $\sim\Lambda$ is:

$$g_l^{-2}\,\mathcal{E}_{\text{link}} \simeq \frac{F^2(\pi\Lambda^2)^2}{2g^2(l)} = \frac{\epsilon(l)}{4} \sim 2.5\ . \tag{63}$$

The number of allowed configurations for the new link is determined by the available directions in coordinate space and group space. Imagining the fluxons to be embedded in a three-dimensional lattice on the scale Λ, there are 5 available directions (this may be an underestimate since no variation was allowed in the fourth direction-fluxons cannot be destroyed!). The available number of directions in group-space will be estimated to be 3 (we may represent the SU(2) sphere by six orthogonal directions and forbid a change of direction by more than 90°). Collecting these estimates and eq. (63) we arrive at the weight of an extra link:

$$W_{\text{link}} \simeq 3 \times 5 \times e^{-2.5} \simeq 1.25 > 1\ . \tag{64}$$

The average length of a fluxon will thus be:

$$\langle L\rangle = \left(\sum_{l=0}^{\infty} W^l\right)^{-1} \sum_{l=0}^{a} l W^l \to \infty \,. \tag{65}$$

We conclude, that within our crude estimate, the instanton background supports a density of random-walking infinitely long fluxons of radius $\sim\Lambda$ (note that four-dimensionally the fluxons represent surfaces!). The random-walk property clearly insures that the fluctuations of the number of fluxons which cut through any given large area will be given by eq. (58), thus leading to quark confinement. An estimate of the average area density μ^2 may be obtained from eq. (65) by dropping the factor 5 (no freedom in directions!) and supplying the relevant length scale:

$$\mu^2 \simeq \frac{3 \times e^{-2.5}}{\pi\Lambda^2} \sim (4\pi\Lambda^2)^{-1} \,. \tag{66}$$

Substituting μ^2 in eq. (60) we find the large distance confining force:

$$\text{force} \simeq \pi/8\Lambda^2 \,. \tag{67}$$

Eqs. (67), (48) determine the hadronic mass scale and provide the connection between the ultra-violet renormalisation group (eq. (7)) and the infra-red bound-state scale μ. As a "phenomenological" aside we observe that if the linear potential is realised by a string which connects two massless quarks we get for the Regge slope,

$$\alpha' = 1\ \text{GeV}^{-2} \simeq \frac{\pi^2}{4\Lambda^2} \,, \tag{68}$$

so that $\Lambda \simeq 0.5$ fm. We also remark that we probably over-estimate the density (and force) since we neglected the interaction of the fluxons with the background "glue" fields. However, we shall not try to improve the above crude considerations at this stage. It should perhaps be stressed that the sheet structure has been used here as a background which forces flux fluctuations to be quantised and provides the necessary space-time scale for their size. Observe that the flux quantisation is characterised by the structure of the centre of the gauge group (in our case SU(2)). Therefore, if SU(3) is the gauge group, there are presumably two kinds of fluxons with strengths $\pm\frac{4}{3}\pi$, thus somewhat reducing the self-action.

We conclude by noting that "integer spin" representations of SU(2) (zero triality for SU(3)) do not feel the confining force since the line integral they generate is not affected by the quantised fluxons. This is in line with the expectation that such external particles should be screened by the vacuum gauge fields *.

* The importance of screening zero-triality external sources was stressed to us by T. Banks.

4. Discussion

Our approach in constructing the vacuum configurations has been to identify the latter with a degenerate set of classical solutions of the equations of motion. Ultra-violet fluctuations were taken into account through renormalisation effects while infra-red fluctuations were treated by explicitly calculating their action.

In assessing the stability of the sheets we face two issues. The easier of the two is the stability of the glue under local fluctuations of the fields on the scale l (the inter-instanton distance). A very crude estimate of these effects may be obtained by computing the extra action needed to bring a local instanton field to zero within a distance of order 2α–3α (in order not to lose the translational entropy). The order of magnitude is easily seen to be $\lesssim \epsilon(\alpha)$.

The only free energy gained by severing the glue is ~ln 6 due to the extra freedom in group-space orientation (3 for directions times 2 for instanton–anti-instanton), which is clearly much smaller than $\epsilon(\alpha) \sim 30$.

The second, much more difficult, issue is the stability under true infra-red fluctuations. Within our framework the question which should be answered is whether other (approximate) solutions of the field equations exist with large entropy (which might not be due to pointlike singularities), and with a totally different global structure (that can overwhelm the contribution of the sheets to the functional integral). We do not have any wisdom concerning this issue which, incidentally, is always present when a self-consistent variational approach is used to approximate a many-body system.

We now list some directions for further investigation:

(i) The Minkowski-space nature of the vacuum configuration and its attendent fluctuations is a subject of considerable conceptual and practical interest. In particular, we believe it is of crucial importance to properly identify those (necessarily rare!) configurations which *do* manage to couple to external quarks. Presumably these should describe an average electric flux trapped inside a tube connecting the $q\bar{q}$ pair, namely an ordered set of fluxons cutting through a sheet which lies in the plane of the space-time loop. However, this is in the nature of a conjecture at this time.

(ii) QCD includes, of course, quark fields so that Dirac fields should be embedded in our vacuum. The resultant average distribution of massless pairs in the self-dual sheet should hopefully lead to a spontaneous breakdown of the SU(2) × SU(2) chiral group and to a massless pion (and a massive η) [14]. More generally, a reliable determination of the vacuum fields and the propagation of fermions would enable a first-principles calculation of the hadronic spectrum to be performed.

(iii) Last, but not least, realistic QCD is an SU(3) (colour) gauge theory and the (hopefully purely technical) extension of $2 \to 3$ should also be faced.

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Appendix

Consider a functional integral

$$Z = \int \prod_{x} \mathrm{d}A(x)\, F[A(x)] \,, \tag{A.1}$$

where x is an index and $A(x)$ are canonical degrees of freedom. Given a configuration which depends upon the parameters $\{\zeta_j\}$:

$$A(x) = a(x, \zeta) + \alpha(x, \zeta) \,. \tag{A.2}$$

Let us follow Faddeev and Popov [7] and multiply eq. (A.1) by a representation of 1:

$$Z = \int \mathrm{d}\zeta\, \mathrm{d}A\, F(a + \alpha)\, \delta[g(A, \zeta)]\, \det\left(\frac{\partial g}{\partial \zeta}\right) . \tag{A.3}$$

We now choose for g the function

$$g_j(A, \zeta) = (a_j | A - a) \,, \tag{A.4}$$

where the notation of eqs. (9), (10) was used. We also define $\mathrm{d}\alpha$ by using an orthonormal ζ-dependent set,

$$\alpha(x, \zeta) = \xi_j\, a_j(x) + \eta(x) \,, \tag{A.5}$$

where η is orthogonal to a_j:

$$(a_j | \eta) = 0 \,. \tag{A.6}$$

Differentiating (A.4) and using the δ-function, we find

$$\frac{\partial g_j}{\partial \zeta_k} = (a_{jk} | \eta) - (a_j | a_k) \,, \tag{A.7}$$

while (A.5) leads to

$$\delta[g]\, \mathrm{d}\alpha = |\det(a_i | a_j)|^{-1/2}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \delta(\xi) \,. \tag{A.8}$$

Collecting (A.7) and (A.8) we arrive at

$$Z = \int \mathrm{d}\zeta\, \mathrm{d}\eta\, |\det(a_i | a_j)|^{-1/2} |\det[(a_i | a_j) - (a_{ij} | \eta)]|\, F(a + \eta) \,, \tag{A.9}$$

which leads the formula used in text upon extracting the factor $\det(a_i | a_j)$ from the second determinant of (A.9)

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