# Application of the nonlinear vector product to Lorentz transformations

Horacio A. Farach, Yakir Aharonov, and Charles P. Poole, Jr.
*Department of Physics, University of South Carolina, Columbia, South Carolina 29208*

Susana I. Zanette
*Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, Brazil*
(Received 1 March 1978; accepted 4 September 1978)

The nonlinear vector product used in a previous publication to treat successive space rotations has been extended to the case of Lorentz transformations by employing an imaginary angle in place of the real angle used for the space rotation case. Expressions were derived for operations of pure Lorentz transformations and for the form of a general Lorentz transformation. The utility of the method was demonstrated by applying it to the derivations of the addition law of velocities and the Thomas precession.

## I. INTRODUCTION

In a previous paper[1] we developed a new nonlinear vector product to describe successive rotations in three dimensions. The rotation vector **R** corresponding to a space rotation through the angle $\phi$ has the following expression:

$$\mathbf{R} = \hat{R}\sin(\phi/2), \tag{1}$$

with its associated scalar

$$r = \cos(\phi/2), \tag{2}$$

where $\hat{R}$ is a unit vector in the direction of the axis of rotation, and

$$R^2 + r^2 = 1. \tag{3}$$

We showed in the previous article that the rotation vector **R** and the scalar $r$ associated with the product of two successive rotation $\mathbf{R}_1$ and $\mathbf{R}_2$ are given by

$$\mathbf{R} = \mathbf{R}_2 \boxtimes \mathbf{R}_1 \tag{4}$$

$$= \mathbf{R}_1 \times \mathbf{R}_2 + \mathbf{R}_1 r_2 + \mathbf{R}_2 r_1, \tag{5}$$

$$r = R_2 \boxdot \mathbf{R}_1 \tag{6}$$

$$= r_1 r_2 - \mathbf{R}_1 \cdot \mathbf{R}_2. \tag{7}$$

These expressions define the square cross product $\boxtimes$ and square dot product $\boxdot$ operators. In this article we will show that the same formalism can be employed to treat the space time rotations of special relativity in which the angle of rotation is imaginary. We will show the utility of this formulation by deriving the addition law for velocities and by calculating the Thomas precession.

## II. LORENTZ TRANSFORMATIONS

A pure Lorentz transformation called a boost involves a transformation from a fixed coordinate system to another coordinate system moving at a uniform velocity **v**. The velocity is customarily written in terms of the reduced velocity $\boldsymbol{\beta}$,

$$\boldsymbol{\beta} = \mathbf{v}/c, \tag{8}$$

where $c$ is the speed of light. The parameter $\gamma$ is defined in terms of the scalar reduced speed $\beta$,

$$\gamma = 1/(1-\beta^2)^{1/2}. \tag{9}$$

During the boost the two coordinate frames retain the orientation of their axes in space, i.e., no space rotation occurs. If a Lorentz transformation involves a simultaneous space rotation it is referred to as a general Lorentz transformation.

By analogy with Eqs. (1) and (2) a pure Lorentz transformation involving a velocity **v** in a direction defined by the unit vector $\hat{B}$ may be characterized by a boost vector **B**,

$$\mathbf{B} = \hat{B}\sin(\psi/2) \tag{10}$$

and an associated boost scalar $b$,

$$b = \cos(\psi/2) \tag{11}$$

with

$$B^2 + b^2 = 1, \tag{12}$$

where the angle $\psi$ is imaginary and may be expressed in terms of its real counterpart $\theta$,

$$\psi = -i\theta. \tag{13}$$

The angles are related to the parameter $\gamma$ through the expressions

$$\sin(\psi/2) = -i\sinh(\theta/2)$$

$$= -i[(\gamma-1)/2]^{1/2}, \tag{14}$$

$$\cos(\psi/2) = \cosh(\theta/2)$$

$$= [(\gamma+1)/2]^{1/2}. \tag{15}$$

Equations (10)–(12) corresponding to this boost are in the same form as Eqs. (1)–(3) for the space rotation. Accordingly they satisfy the nonlinear vector product formalism developed in the previous publication.[1]

## III. SUCCESSIVE LORENTZ TRANSFORMATIONS

It is well known that the product of two successive noncolinear boosts is not a boost but rather it is the product of a space rotation and a boost. Accordingly we can write the square cross and square dot products for the transformation

$$\mathbf{B}_2 \boxtimes \mathbf{B}_1 = \mathbf{R} \boxtimes \mathbf{B}, \tag{16}$$

$$\mathbf{B}_2 \boxdot \mathbf{B}_1 = \mathbf{R} \boxdot \mathbf{B}. \tag{17}$$

 0002-9505/79/030247-03$00.50  

Using Eqs. (4)–(7) these expressions may be expanded to the form

$$\mathbf{B}_1 \times \mathbf{B}_2 + \mathbf{B}_1 b_2 + \mathbf{B}_2 b_1 = \mathbf{B} \times \mathbf{R} + \mathbf{R}b + \mathbf{B}r, \quad (18)$$

$$b_1 b_2 - \mathbf{B}_1 \cdot \mathbf{B}_2 = rb - \mathbf{R} \cdot \mathbf{B}. \quad (19)$$

The real and imaginary parts of Eq. (18) can be equated separately to give

$$\mathbf{B}_1 \times \mathbf{B}_2 = \mathbf{R}b, \quad (20)$$

$$\mathbf{B}_1 b_2 + \mathbf{B}_2 b_1 = \mathbf{B} \times \mathbf{R} + \mathbf{B}r. \quad (21)$$

From Eq. (20) we deduce

$$\mathbf{R} \cdot \mathbf{B}_1 = 0 \quad (22)$$

and

$$\mathbf{R} \cdot \mathbf{B}_2 = 0 \quad (23)$$

and if we form the dot product with $\mathbf{R}$ of both sides of Eq. (21) we conclude that

$$\mathbf{R} \cdot \mathbf{B} = 0. \quad (24)$$

Thus we see that the final boost vector $\mathbf{B}$ lies in the same plane as the boost vectors $\mathbf{B}_1$ and $\mathbf{B}_2$, and that the rotation vector $\mathbf{R}$ is perpendicular to this plane.

Equation (20) can be written

$$\mathbf{R} = \mathbf{B}_1 \times \mathbf{B}_2 / b \quad (25)$$

and the scalar product of Eq. (21) with itself gives

$$(\mathbf{B}_1 b_2 + \mathbf{B}_2 b_1)^2 = R^2 B^2 + B^2 r^2, \quad (26)$$

which in turn may be simplified, using Eq. (3),

$$B^2 = (\mathbf{B}_1 b_2 + \mathbf{B}_2 b_1)^2. \quad (27)$$

We may define the vector on the right hand side of this equation as $\mathbf{B}_p$,

$$\mathbf{B}_p = \mathbf{B}_1 b_2 + \mathbf{B}_2 b_1 \quad (28)$$

where $\mathbf{B}$ and $\mathbf{B}_p$ have the same magnitude. If we form the vector product of $\mathbf{R}$ with Eq. (21) and the scalar product of $r$ with the same equation and add the two expressions, we obtain

$$\mathbf{B} = \mathbf{R} \times \mathbf{B}_p + \mathbf{B}_p r. \quad (29)$$

It is easy to show by direct calculation that $r$ equals the scalar product of unit vectors $\mathbf{B}$ and $\mathbf{B}_p$ in the $\mathbf{B}$ and $\mathbf{B}_p$ directions, respectively,

$$r = \mathbf{B} \cdot \mathbf{B}_p. \quad (30)$$

Finally, using the definition of the square cross product and Eq. (28), we can write

$$\mathbf{B}_2 \times \mathbf{B}_1 = \mathbf{R}\mathrm{b} + \mathbf{B}_p, \quad (31)$$

which corresponds to setting

$$\mathbf{R} = (\mathbf{B}_2 \times \mathbf{B}_1)_\perp / b, \quad (32)$$

$$\mathbf{B}_p = (\mathbf{B}_2 \times \mathbf{B}_1)_\parallel, \quad (33)$$

where $\perp$ and $\parallel$ are relative to the plane of $\mathbf{B}_1$ and $\mathbf{B}_2$.

## IV. ADDITION OF VELOCITIES

To obtain a general expression for the velocity $\boldsymbol{\beta}$ which is the sum of the two velocities $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ we can note from Eqs. (10), (11), (14), and (15) that

$$b\mathbf{B} = -(\tfrac{1}{2}) i \boldsymbol{\beta} \gamma, \quad (34)$$

$$b^2 - B^2 = \gamma. \quad (35)$$

Therefore we have

$$\boldsymbol{\beta} = 2i[b\mathbf{B}/(b^2 - B^2)], \quad (36)$$

where $b$ and $\mathbf{B}$ are given in terms of $\mathbf{B}_1$, $b_1$, $\mathbf{B}_2$, and $b_2$ by Eqs. (28) and (31).

The quantity $\gamma$ for the sum of the velocities may be derived from Eq. (27) which may be written

$$B^2 = B_1^2 b_2^2 + B_2^2 b_1^2 + 2 b_1 b_2 B_1 B_2 \cos\alpha. \quad (37)$$

Using Eqs. (10)–(15), we may transform this expression to the form

$$\gamma = \gamma_+ \cos^2(\alpha/2) + \gamma_- \sin^2(\alpha/2), \quad (38)$$

where, from Eq. (9),

$$\gamma_\pm = (1 - \beta_\pm^2)^{-1/2} \quad (39)$$

and

$$\beta_\pm = (\beta_1 \pm \beta_2)/(1 \pm \beta_1 \beta_2), \quad (40)$$

where $\beta_+$ and $\beta_-$ are the velocities obtained for the parallel and antiparallel addition of $\beta_1$ and $\beta_2$. Thus we have $\gamma$ expressed as a function of $\gamma_+$ and $\gamma_-$ which are the $\gamma$'s associated with the parallel and antiparallel addition of velocities. In other words for two arbitrary velocities which subtend on arbitrary angle $\alpha$ the final $\gamma$ can be obtained from the sums and differences of their magnitudes. The direction is given by Eq. (29) or (36).

## V. THOMAS PRECESSION

To derive the Thomas precession from our formalism, we follow Jackson.[2] We start with his Eq. (11.49),

$$K' \dashv\!\boxed{-\mathbf{v}}\!\to K \dashv\!\boxed{\mathbf{v} + \delta\mathbf{v}}\!\to K'', \quad (41)$$

which corresponds to a boost from the rest system $K'$ of a particle moving at a velocity $\mathbf{v}$ to the laboratory system $K$ followed by a boost back to the rest system $K''$ of the particle which is now moving at the velocity $\mathbf{v} + \delta\mathbf{v}$.

The first Lorentz transformation is given by $\mathbf{B}_1$,

$$\mathbf{B}_1(-\mathbf{v}) = \hat{\mathbf{B}}_1 \sin(\psi/2). \quad (42)$$

The second Lorentz transformation is given by $\mathbf{B}_2$,

$$\mathbf{B}_2(\mathbf{v} + \delta\mathbf{v}) = \hat{\mathbf{B}}_2 \sin(\psi/2 + \delta\psi/2), \quad (43)$$

where

$$\hat{\mathbf{B}}_1 = -\boldsymbol{\beta}/\beta, \quad (44)$$

$$\hat{\mathbf{B}}_2 = (\boldsymbol{\beta} + \delta\boldsymbol{\beta})/|\boldsymbol{\beta} + \delta\boldsymbol{\beta}| \sim (\boldsymbol{\beta} + \delta\boldsymbol{\beta})/\beta, \quad (45)$$

and

$$\sin(\psi/2 + \delta\psi/2) \approx \sin(\psi/2). \quad (46)$$

For small $\alpha$'s, where $\alpha$ is the angle between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$, it can be shown, using Eq. (28), that

$$b \approx 1. \quad (47)$$

Therefore the rotation vector of Eq. (25) becomes

$$\mathbf{R} = [(\gamma - 1)/2](\beta \times \delta\beta/\beta^2), \tag{48}$$

since $\beta \times \beta = 0$ and use was made of Eqs. (10) and (14). But $\mathbf{R}$ is a rotation vector so it can be written

$$\mathbf{R} = \hat{R}\sin(\delta\theta/2). \tag{49}$$

For sufficiently small $\delta v$ this can be approximated by

$$\mathbf{R} \approx \delta\theta/2. \tag{50}$$

Comparing Eqs. (48) and (50) permits us to write

$$\delta\theta = [(\gamma - 1)/\beta^2])\beta \times \delta\beta. \tag{51}$$

This may be expressed in terms of an angular velocity $\omega_T$ and an acceleration $\mathbf{a}$,

$$\omega_T = d\theta/dt, \tag{52}$$

$$\mathbf{a} = d\mathbf{v}/dt,$$

to give the usual Thomas precession formula

$$\omega_T = -[\gamma^2/(\gamma + 1)]/(\mathbf{v} \times \mathbf{a}/c^2),$$

where from Eq. (9) $\beta^2$ is replaced by $(\gamma^2 - 1)/\gamma^2$. For nonrelativistic velocities $\gamma \sim 1$ and we obtain the Thomas precession expression often quoted in modern physics texts

$$\omega_T = -\mathbf{v} \times \mathbf{a}/2c^2. \tag{53}$$

Ashworth[3] recently published an alternate method to obtain the Thomas precession. Instead of considering Lorentz transformations between three inertial frames of reference, as is done in typical texts[2] and by us above, Ashworth made use of a single Lorentz transformation between two inertial frames with a relative velocity that is not parallel to any of the coordinate axes.

## VI. CONCLUSIONS

In this article we extended the formalism developed in the previous paper on space rotations to the case of space time rotations. The form of the boost vector of a pure Lorentz transformation was arrived at by using a purely imaginary angle in the expression deduced previously for space rotations. The operations involved in the application of successive boosts were deduced and the form of a general Lorentz transformation was found to separate naturally into boost-type and space rotation-type parts. The formalism was applied to derive expressions for $\beta$ and $\gamma$ in the addition of velocities and for the rotation corresponding to the Thomas precession.


[1]Y. Aharonov, H. A. Farach, and C. P. Poole, Am. J. Phys. **45**, 451 (1977).

[2]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[3]D. G. Ashworth, Il Nuovo Cimento B **40**, 242 (1977).