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A Vector Product Formulation of Special Relativity and Electromagnetism

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The vector product method developed in previous articles for space rotations and Lorentz transformations is extended to the cases of four-vectors, antisymmetric tensors, and their transformations in Minkowski space. The electromagnetic fields are expressed in "six-vector" form using the notation $\mathbf{H} + i\mathbf{E}$, *and this vector form is shown to be relativistically invariant. The wave equations of electromagnetism are derived using these vector products. The following three equations are deduced, which summarize electrodynamics in a compact form:* (1) *Maxwell's four equations expressed as one,* (2) *the scalar and vector potential wave equations combined into one relation, and* (3) *the wave equations for the electric and magnetic fields and the continuity equation combined together. Space inversion, time reversal, and magnetic monopoles are also treated.*

1. INTRODUCTION

In the first article of this series,[1] which we shall refer to as I, we expressed a rotation through an angle ϕ in the direction of a unit vector $\hat{n}$ in terms of a rotation vector **R**,

$$\mathbf{R} = \hat{n} \sin \tfrac{1}{2}\phi \tag{1}$$

and an associated scalar r,

$$r = (1 - R^2)^{1/2} = \cos \tfrac{1}{2}\phi \tag{2}$$

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We showed that two successive rotations $\mathbf{R}_1$ and $\mathbf{R}_2$ produce a third rotation $\mathbf{R}_3$ with their vector and scalar parts given by the nonlinear vector product operations $\boxtimes$ and $\boxdot$ defined by

$$\mathbf{R}_3 = \mathbf{R}_2 \boxtimes \mathbf{R}_1 = \mathbf{R}_1 \times \mathbf{R}_2 + r_1\mathbf{R}_2 + r_2\mathbf{R}_1 \tag{3}$$

$$r_3 = \mathbf{R}_2 \boxdot \mathbf{R}_1 = r_1 r_2 - \mathbf{R}_1 \cdot \mathbf{R}_2 \tag{4}$$

Several examples of the utility of this approach to space rotations were presented in I.

In the second article,[2] which we shall call II, we showed that the spacetime rotation called a pure Lorentz transformation or boost has the same form as Eqs. (1) and (2) with the angle ϕ having an imaginary value $\phi = -i\psi$. This converts the sines and cosines of Eqs. (1) and (2) to their hyperbolic counterparts, which are related to the quantity $\gamma = (1 - \beta^2)^{-1/2}$ as follows:

$$\sinh\frac{1}{2}\psi = \left(\frac{\gamma - 1}{2}\right)^{1/2} \qquad \cosh\frac{1}{2}\psi = \left(\frac{\gamma + 1}{2}\right)^{1/2} \tag{5}$$

In II the usefulness of the nonlinear product method was illustrated by applying it to the determination of the addition law of velocities and the Thomas precession.

There is an isomorphism between the nonlinear vector products $\boxtimes$ and $\boxdot$ of Eqs. (1)–(2) and a linear vector product denoted by the symbol $\square$

$$R_3 = R_2 \square R_1 \tag{6}$$

of associated four-component real vectors $(R_{ix}, R_{iy}, R_{iz}, r_i)$ whose first three components form the vectors $\mathbf{R}_i$ of Eq. (1) and whose fourth component is the scalar r_i of Eq. (2). The linearity of the resulting vector product may be found by writing Eqs. (3) and (4) in the form

$$\begin{aligned} R_{3i} &= R_{1j}R_{2k} - R_{1k}R_{2j} + r_1 R_{2i} + R_{1i} r_2 \\ r_3 &= r_1 r_2 - R_{1i}R_{2i} - R_{1j}R_{2j} - R_{1k}R_{2k} \end{aligned} \qquad i, j, k = x, y, z \text{ cyclically} \tag{7}$$

The linear vector product operation $\square$ of Eq. (7) preserves the product of the magnitudes M_1, M_2, and M_3 associated with these four-component vectors

$$M_3 = M_1 M_2 \tag{8}$$

where in this case $M_i = R_i^2 + r_i^2 = 1$ for $i = 1, 2, 3$.

If the normalization condition $R^2 + r^2 = 1$ and the reality condition on the vectors R are relaxed, then we can define a vector V with three real components v_x, v_y, v_z which form the Cartesian vector **V** and the fourth imaginary component iv_t

$$V = \{v_x, v_y, v_z, iv_t\} \tag{9}$$

This may be written in condensed notation as

$$V = \{\mathbf{V}, iv\} \tag{10}$$

The fact that **V** is boldface indicates a quantity which transforms like a three-vector under space rotations. Associated with every four-vector V there is a conjugate four vector V^* defined by

$$V^* = \{\mathbf{V}, iv\}^* = \{\mathbf{V}, -iv\} \tag{11}$$

Two four-vectors $\{\mathbf{V}_1, iv_1\}$ and $\{\mathbf{V}_2, iv_2\}$ have the scalar product

$$V_1 \cdot V_2 = \sum_{i=1}^{4} (v_{1i}v_{2i}) \tag{12}$$

and by convention a four-vector is called spacelike, lightlike, or timelike depending upon the sign of its magnitude or self scalar product $V \cdot V = V^2 - v^2$:

$$V^2 - v^2 \begin{cases} > 0 & \text{spacelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{timelike} \end{cases} \tag{13}$$

In the present article we will start with the four-vectors as primitive, and using the linear vector product operation $\square$ on alternating four-vectors and conjugate four-vectors, we will construct the higher order spacetime vectors $V_1 \square V_2^*$, $V_1 \square V_2^* \square V_3$, and $V_1 \square V_2^* \square V_3 \square V_4^*$, which we refer to respectively as relativistic vectors of second order, third order, and fourth order. We will show that space rotations and boosts are second-order and more general Lorentz transformation are fourth-order relativistic vectors. Antisymmetric second-rank tensors appear in the formalism as second-order relativistic vectors. The fundamental equations of electromagnetism will be written in terms of vectors of these various orders.

The treatment presented in this article is closely related to Clifford algebras[3,4] which were discovered a hundred years ago and which have been extensively used by Hestenes[5,6] in recent years. At the end of the article we will make some comments about Clifford algebras.

2. RELATIVISTIC VECTORS

A four-vector is called a first-order relativistic vector. The $\boxtimes$ and $\boxdot$ operations defined by Eqs. (3) and (4) may be employed to form a second-order relativistic vector D from a four-vector $V_1 = \{\mathbf{V}_1, iv_1\}$ and a complex conjugate four-vector $V_2^* = \{\mathbf{V}_2, -iv_2\}$. Thus we write

$$D = V_1 \square V_2^* = [\mathbf{D}, d] \tag{14}$$

where D has a complex space part

$$\mathbf{D} = \mathbf{V}_1 \boxtimes \mathbf{V}_2^* = -\mathbf{V}_1 \times \mathbf{V}_2 + i(v_1\mathbf{V}_2 - \mathbf{V}_1 v_2) \tag{15}$$

and a real time part

$$d = \mathbf{V}_1 \boxdot \mathbf{V}_2^* = -\mathbf{V}_1 \cdot \mathbf{V}_2 + v_1 v_2 \tag{16}$$

The conjugate second-order relativistic vector corresponding to D is

$$D^* = V_1^* \square V_2 = [\mathbf{D}^*, d] \tag{17}$$

In general both D and D^* have complex magnitudes.

A third-order relativistic vector Q is generated from three first-order vectors V_1, V_2^*, V_3 as follows:

$$Q = V_1 \square V_2^* \square V_3 = \{\mathbf{V}_1, iv_1\} \square \{\mathbf{V}_2, -iv_2\} \square \{\mathbf{V}_3, iv_3\} = \{\mathbf{Q}, q\} \tag{18}$$

where $\mathbf{Q}$ and q are easily found to be (cf. the Appendix)

$$\begin{aligned} \mathbf{Q} &= -\mathbf{V}_1(\mathbf{V}_2 \cdot \mathbf{V}_3) + \mathbf{V}_2(\mathbf{V}_3 \cdot \mathbf{V}_1) - \mathbf{V}_3(\mathbf{V}_1 \cdot \mathbf{V}_2) + v_1 v_2 \mathbf{V}_3 + v_2 v_3 \mathbf{V}_1 \\ &\quad - v_3 v_1 \mathbf{V}_2 + i[v_1 \mathbf{V}_2 \times \mathbf{V}_3 + v_2 \mathbf{V}_3 \times \mathbf{V}_1 + v_3 \mathbf{V}_1 \times \mathbf{V}_2] \\ q &= \mathbf{V}_2 \cdot (\mathbf{V}_3 \times \mathbf{V}_1) + i[v_1 v_2 v_3 - v_1(\mathbf{V}_2 \cdot \mathbf{V}_3) + v_2(\mathbf{V}_3 \cdot \mathbf{V}_1) - v_3(\mathbf{V}_1 \cdot \mathbf{V}_2)] \end{aligned} \tag{19}$$

This triple product result can aid in evaluating transformations of four-vectors which appear later. Another useful way to write Eq. (18) is

$$Q = V_1 \square D^* \tag{20}$$

A third-order relativistic vector Q can be written uniquely as the sum

$$Q = V + V_p \tag{21}$$

of a four-vector V and what may be called a pseudo-four-vector V_p

$$V = \{\mathbf{V}, iv\}, \qquad V_p = \{i\mathbf{V}_p, -v_p\} \tag{22}$$

where $\mathbf{V}$, v, $\mathbf{V}_p$, and v_p are all real. Both V and V_p have the same transformation properties, to be discussed below. The terminology pseudo-four-vector for V_p arises from its behavior under the parity operation, as will be explained later [cf. Eq. (89)].

A fourth-order relativistic vector H

$$H = V_1 \Box V_2^* \Box V_3 \Box V_4^* = [\mathbf{H}, h] \tag{23}$$

has complex vector and scalar parts $\mathbf{H}$ and h, respectively. It may also be considered as the linear vector product of two second-order relativistic vectors

$$H = D_1 \Box D_2 \tag{24}$$

where use was made of Eq. (14).

A fourth-order relativistic vector H may be written uniquely as the sum

$$H = S + s + s_p \tag{25}$$

of a six-vector S, a scalar s, and what we will refer to as a pseudoscalar s_p

$$S = [\mathbf{S}' + i\mathbf{S}'', 0] \tag{26a}$$

$$s = [0, s] \tag{26b}$$

$$s_p = [0, is_p] \tag{26c}$$

where $\mathbf{S}'$, $\mathbf{S}''$, s, and s_p are all real. All three quantities S, s, and s_p have the same transformation law, as will be explained later.

In summary, we have shown that the present formalism contains four orders of relativistic vectors $(\mathbf{G}' + i\mathbf{G}'', ig' + g'')$. There are two first-order vectors, namely a vector $\{\mathbf{G}', ig'\}$ and a pseudovector $\{i\mathbf{G}'', -g''\}$ each of which has four parameters G_x', G_y', G_z', g' and G_x'', G_y'', G_z'', g'', respectively. A second-order relativistic vector with the form $[\mathbf{G}' + i\mathbf{G}'', g'']$ has seven parameters, and a special type of second-order vector called a six-vector $[\mathbf{G}' + i\mathbf{G}'', 0]$ has only six parameters. Third- and fourth-order relativistic vectors are each of the form $(\mathbf{G}' + i\mathbf{G}'', ig' + g'')$ and have all eight parameters. The formation of higher order vector products

$$V_1 \Box V_2^* \Box V_3 \Box V_4^* \Box V_5 \cdots$$

does not increase the number of parameters and hence does not generate new orders of vectors. They remain third or fourth order, depending upon whether or not they are even or odd in the number of primitives which form them. Therefore first- and second-order forms may be considered as special cases and their third- and fourth-order counterparts as general cases, respectively, of odd and even relativistic vectors.

In an alternate approach, pseudovectors may be selected as primitives and D may be derived from the product $-V_{p1} \Box V_{p2}^*$ of a pseudovector and a conjugate pseudovector. The higher order relativistic vectors may be constructed analogously.

3. PROPERTIES OF TRANSFORMATIONS

A transformation $T = (\mathbf{T}, t)$ is an even-order normalized relativistic vector. Some of the properties of particular transformations were given in I and II and are summarized in Table I. In the present section we will show how space rotation and boost transformations can be derived from primitive four-vectors, and then we will discuss some of the general properties of transformations.

A space rotation $\mathbf{R}$ may be considered as formed via Eq. (14) from two spacelike four-vectors V_1 and V_2^* with magnitudes of $+1$,

$$V_1 = \{-\hat{n}_1, 0\}, \qquad V_2^* = \{\hat{n}_2, 0\} \tag{27}$$

where $\hat{n}_1$ and $\hat{n}_2$ are unit space vectors which subtend the angle $\phi/2$ of Eqs. (1) and (2). This gives

$$R = [\hat{n}_1 \times \hat{n}_2, \hat{n}_1 \cdot \hat{n}_2] = [\mathbf{R}, r]$$

where use was made of Eqs. (15) and (16). A boost B is derived from two timelike four-vectors

$$V_1 = \{0, i\}, \qquad V_2^* = \{\mathbf{B}, ib\} \tag{28}$$

with magnitudes of -1, where

$$B^2 - b^2 = -1$$

Making use of Eq. (14) gives

$$B = [i\mathbf{B}, b] \tag{29}$$

Table I. Properties of Proper Transformations[a]

Transformation	$\mathbf{T}$	t	$t^2 + T^2 = 1$	$t^2 - T^2$	$2tT$
Identity $[0, 1]$	0	1	$t^2 = 1$	1	0
Space rotation $[\mathbf{R}, r]$	$\mathbf{R} = \hat{R} \sin \frac{1}{2}\theta$	$r = \cos \frac{1}{2}\theta$	$r^2 + R^2 = 1$	$r^2 - R^2 = \cos \theta$	$2rR = \sin \theta$
Boost $[i\mathbf{B}, b]$	$i\mathbf{B} = i\hat{\beta} \sinh \frac{1}{2}\theta = i\hat{\beta}\left(\frac{\gamma - 1}{2}\right)^{1/2}$	$b = \cosh \frac{1}{2}\theta = \left(\frac{\gamma + 1}{2}\right)^{1/2}$	$b^2 - B^2 = 1$	$b^2 + B^2 = \gamma$	$2bB = \beta\gamma$
Two successive boosts					
$[i\mathbf{B}_2, b_2] \square [i\mathbf{B}_1, b_1]$	$\mathbf{B}_1 \times \mathbf{B}_2 + i(b_1\mathbf{B}_2 + b_2\mathbf{B}_1)$	$b_1 b_2 + \mathbf{B}_1 \cdot \mathbf{B}_2$	$(b_1^2 - B_1^2)(b_2^2 - B_2^2) = 1$	—	—
$= [\mathbf{R}, r] \square [i\mathbf{B}, b]$	$b\mathbf{R} + i(r\mathbf{B} + \mathbf{B} \times \mathbf{R})$	rb	$(r^2 + R^2)(b^2 - B^2) = 1$	—	—
General Lorentz					
$[L, l] = [\mathbf{R}, r] \square [i\mathbf{B}, b]$	$b\mathbf{R} + i(r\mathbf{B} + \mathbf{B} \times \mathbf{R})$	$rb - i\mathbf{R} \cdot \mathbf{B}$	$L \cdot L + l^2 = 1$	—	—

[a] Two forms of the product of successive boosts are given. The values of $t^2 - T^2$ and $2tT$ are omitted for the last three transformations because of their complexity. The notation $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = (1 - \beta^2)^{-1/2}$ is used. For details about the entries in this table, see I and II.

which is the boost vector given in II and listed in Table I. Hence the basic space rotation and boost transformations are second-order relativistic vectors. We see from these equations that a pure rotation is real and a boost has a real time part and a purely imaginary space part.

In an alternate approach the pseudovectors $\{i\hat{n}_1, 0\}$ and $\{i\hat{n}_2, 0\}$ may be used to form a space rotation.

The most general proper, or orthochronous, Lorentz transformation L has complex space and time parts and may be written uniquely as the product of a space rotation R and a boost $B^{(7)}$

$$[\mathbf{L}, l] = [\mathbf{R}, r] \,\square\, [i\mathbf{B}, b] = [b\mathbf{R} + i(r\mathbf{B} + \mathbf{B} \times \mathbf{R}), rb - i\mathbf{R} \cdot \mathbf{B}] \qquad (30)$$

or in a more condensed notation

$$L = R \,\square\, B \qquad (31)$$

This makes it a fourth-order relativistic vector. In II we treated the special case where the Lorentz transformation is dervived from two successive noncolinear boosts and we showed that $\mathbf{R}$ is perpendicular to $\mathbf{B}$ so that $l = rb$ is purely real. The magnitude $L^2 + l^2$ of L is of course unity

$$L \cdot L = L^2 + l^2 = 1 \qquad (32)$$

as may be demonstrated by direct calculation from Eq. (12).

In general the magnitude of a transformation is ± 1 through the normalization condition

$$T \cdot T = T^2 + t^2 = \pm 1 \qquad (33)$$

If the magnitude is $+ 1$, the transformation is a proper space–time rotation, and such transformations are the subject matter of this section. We will see later that a magnitude of $- 1$ corresponds to an improper space–time rotation which includes a space–time inversion ($\mathbf{r} \to -\mathbf{r}$, $t \to -t$).

There is a unit transformation U

$$U = [0, 1] \qquad (34)$$

which commutes with all relativistic vectors G and leaves them unchanged,

$$U \,\square\, G = G \,\square\, U = G \qquad (35)$$

To obtain the reciprocal $T^{-1} = T' = (\mathbf{T}', t')$ of the transformation T

$$T \,\square\, T^{-1} = T^{-1} \,\square\, T = U \qquad (36)$$

we combine the scalar part $tt' - \mathbf{T} \cdot \mathbf{T}' = 1$ of the expression $T \square T' = U$ with the normalization condition (32) to give

$$t(t' \mp t) = \mathbf{T} \cdot (\mathbf{T}' \pm \mathbf{T}) \tag{37}$$

The upper signs apply to a proper Lorentz transformation, so that $t' = t$ and $\mathbf{T}' = -\mathbf{T}$. Therefore, the reciprocal of $[\mathbf{T}, t]$ is $[-\mathbf{T}, t]$:

$$T^{-1} = [\mathbf{T}, t]^{-1} = [-\mathbf{T}, t] \tag{38}$$

The lower signs apply to improper Lorentz transformations, as will be explained later. The reciprocal of the product $T_1 \square T_2$ of two transformations T_1 and T_2 is

$$(T_1 \square T_2)^{-1} = T_2^{-1} \square T_1^{-1} \tag{39}$$

4. TRANSFORMING RELATIVISTIC VECTORS

A four-vector V in a particular space–time coordinate system can be transformed to the form V' in another space–time coordinate system by means of a proper transformation T through the following operation:

$$V' = T \square V \square T^{*-1} \tag{40}$$

The proof for this will be presented in the next section. The transformation of the conjugated counterpart V^* of V is obtained by taking the complex conjugate of both sides of this expression

$$V'^* = T^* \square V^* \square T^{-1} \tag{41}$$

In like manner we have for a pseudo-four-vector

$$V_p' = T \square V_p \square T^{*-1}, \qquad V_p'^* = T^* \square V_p^* \square T^{-1} \tag{42}$$

Thus we see that a general third-order relativistic vector Q is composed of vector and pseudovector parts which do not mix together under a transformation. The second-order relativistic vector $D = [\mathbf{D}' + i\mathbf{D}'', d + id_p]$ of Eq. (14) transforms as follows:

$$D' = (T \square V \square T^{*-1}) \square (T^* \square V^* \square T^{-1}) = T \square D \square T^{-1} \tag{43}$$

where D' has the form

$$D' = [\mathbf{D}''' + i\mathbf{D}'''', d + id_p] \tag{44}$$

and the scalar parts id and d_p are uneffected by the transformation. This is the justification for dividing D' into the six-vector S, scalar s, and pseudoscalar s_p parts in accordance with Eq. (44). The three parts do not mix under transformations.

The preceding expressions may be generalized to give for odd-order G_{odd} and even-order G_{even} relativistic vectors, respectively,

$$G_{\text{odd}'} = T \,\Box\, G_{\text{odd}} \,\Box\, T^{*-1} \tag{45a}$$

$$G_{\text{even}'} = T \,\Box\, G_{\text{even}} \,\Box\, T^{-1} \tag{45b}$$

To clarify the difference between these two expressions we consider the case in which the space part $\mathbf{T}$ of T is in the z direction, and we write for the corresponding transformation of a general high-order relativistic vector G

$$T \,\Box\, G \,\Box\, T' = [T_z, t] \,\Box\, (\mathbf{G}' + i\mathbf{G}'', -g'' + ig') \,\Box\, [T_z', t'] \tag{46}$$

and T' is T^{*-1} or T^{-1}, depending upon the order of G

$$[T_z', t'] = [-T_z^*, t^*] \qquad \text{odd order } G \tag{47a}$$

$$[T_z', t'] = [-T_z, t] \qquad \text{even order } G \tag{47b}$$

Table II illustrates how the components of G mix when T is a space rotation and a boost for both odd and even orders of G. We see from the table that the components mix in pairs, and the pairs are either x and y pairs or z and time pairs. A fourth possibility given in the last row does not occur in special relativity. The table illustrates the particular case in which T is along one coordinate axis. More general orientations of T produce more mixing of components.

It was mentioned above that the magnitude of a relativistic vector $G = (\mathbf{G}' + i\mathbf{G}'', ig' - g'')$ given by

$$\mathbf{G} \cdot \mathbf{G} = G'^2 - G''^2 + g''^2 - g'^2 + 2i(\mathbf{G}' \cdot \mathbf{G}'' - g'g'')$$

is not changed by a transformation. Therefore its real and imaginary parts separately remain invariant. As a result the invariants of a first-order relativistic vector $\{\mathbf{G}', ig'\}$ and pseudovector $\{i\mathbf{G}'', -g''\}$ are, respectively, $G'^2 - g'^2$ and $-G''^2 + g''^2$. A second-order relativistic six-vector $[\mathbf{G}' + i\mathbf{G}'', 0]$ possesses two invariants $G'^2 - G''^2$ and $\mathbf{G}' \cdot \mathbf{G}''$. The overall third-order relativistic vector $\{\mathbf{G}' + i\mathbf{G}'', ig' - g''\}$ and its four-vector $\{\mathbf{G}', ig''\}$ and four-pseudovector $\{i\mathbf{G}'', -g''\}$ components (21) and (22), respectively, are individually preserved in magnitude and hence there are three associated

Table II. How the Components of a General Relativistic Vector $G = (\mathbf{G}' + i\mathbf{G}'', g'' + ig')$ Are Mixed As Linear Combinations by Four Types of z-Direction Transformations $(\mathbf{T}_z, t)$ □ $(\mathbf{G}, g)$ □ $[\mathbf{T}_z', t]$

Transformation	$[\mathbf{T}, t]$	$[\mathbf{T}', t]$	Components mixed in pairs	Components unmixed	Order of $(\mathbf{G}, g)$
Space rotation	$[\mathbf{R}, r]$	$[-\mathbf{R}, r]$	$(G_x', G_y'), (G_x'', G_y'')$	G_z', g', G_z'', g''	Odd or even
Boost	$[i\mathbf{B}, b]$	$[i\mathbf{B}, b]$	$(G_z', g'), (G_z'', g'')$	G_x', G_y', G_x'', G_y''	Odd
Boost	$[i\mathbf{B}, b]$	$[-i\mathbf{B}, b]$	$(G_x', G_y''), (G_x'', G_y')$	$G_z' g', G_z'', g''$	Even
Space rotation	$[\mathbf{R}, r]$	$[\mathbf{R}, r]$	$(G_z', g''), (G_z'', g)$	G_x', G_y', G_x'', G_y''	No known application

invariants, namely $G'^2 - g'^2$, $-G''^2 + g''^2$, and $\mathbf{G}' \cdot \mathbf{G}'' - g'g''$. A fourth-order relativistic vector $[\mathbf{G}' + i\mathbf{G}'', ig' + g'']$ and its six-vector $[\mathbf{G}' + i\mathbf{G}'', 0]$, scalar $[0, g'']$ and pseudoscalar $[0, ig']$ parts, respectively, are individually conserved and hence its four invariants $G'^2 - G''^2$, g'^2, g''^2, and $\mathbf{G}' \cdot \mathbf{G}''$ are individually preserved.

A space–time transformation L, being an even-order relativistic vector, transforms to a new system via Eq. (45b). For example, in this equation let G_{even} be the boost $[i\mathbf{B}, b]$ and T be the boost $[i\mathbf{B}_T, b_T]$, which gives

$$[i\mathbf{B}', b'] = [\gamma_T(\boldsymbol{\beta}_T \times \mathbf{B}) + i\gamma_T\mathbf{B} + i(1 - \gamma_T)\hat{\beta}_T(\hat{\beta}_T \cdot \mathbf{B}), b] \tag{48}$$

where $\gamma_T = (1 - \beta_T^2)^{-1/2}$ and $\boldsymbol{\beta}_T$ is the reduced velocity. The fact that the time part $b = b'$ remains unchanged shows that the transformed boost $B' = B$ retains the magnitude of its original velocity; only its direction changes.

In this section we merely stated the transformation properties of relativistic vectors without proof. In the next section we will prove them for a four-vector transformed by a space rotation and by a boost.

5. TRANSFORMING FOUR-VECTORS

We mentioned above that the law of transformation for a four-vector V is

$$V' = T \square V \square T^{*-1} \tag{49}$$

We will show that this is valid by writing down the results of this transformation for the cases of a space rotation and a boost.

For a space rotation Eq. (49) has the form

$$\{\mathbf{V}', iv'\} = [\mathbf{R}, r] \square \{\mathbf{V}, iv\} \square [-\mathbf{R}, r] \tag{50}$$

and when we work out the triple products [cf. Eqs. (19)] we obtain

$$\mathbf{V}' = 2r\mathbf{V} \times \mathbf{R} + (r^2 - R^2)\,\mathbf{V} + 2\mathbf{R}(\mathbf{V} \cdot \mathbf{R}) \tag{51}$$

If $\mathbf{V}$ is written in terms of components parallel and perpendicular to the rotation direction and use is made of Table I, we arrive at the more familiar expressions

$$V_{\parallel}' = V_{\parallel}, \qquad \mathbf{V}_{\perp}' = (\sin\theta)\,\mathbf{V} \times \hat{R} + (\cos\theta)\,\mathbf{V}_{\perp}, \qquad v' = v \tag{52}$$

These constitute the well-known formulas for a space rotation.

invariants, namely $G'^2 - g'^2$, $- G''^2 + g''^2$, and $\mathbf{G}' \cdot \mathbf{G}'' - g'g''$. A fourth-order relativistic vector $[\mathbf{G}' + i\mathbf{G}'', ig' + g'']$ and its six-vector $[\mathbf{G}' + i\mathbf{G}'', 0]$, scalar $[0, g'']$ and pseudoscalar $[0, ig']$ parts, respectively, are individually conserved and hence its four invariants $G'^2 - G''^2$, g'^2, g''^2, and $\mathbf{G}' \cdot \mathbf{G}''$ are individually preserved.

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$$[i\mathbf{B}', b'] = [\gamma_T(\boldsymbol{\beta}_T \times \mathbf{B}) + i\gamma_T\mathbf{B} + i(1 - \gamma_T)\hat{\beta}_T(\hat{\beta}_T \cdot \mathbf{B}), b] \tag{48}$$

where $\gamma_T = (1 - \beta_T{}^2)^{-1/2}$ and $\boldsymbol{\beta}_T$ is the reduced velocity. The fact that the time part $b = b'$ remains unchanged shows that the transformed boost $B' = B$ retains the magnitude of its original velocity; only its direction changes.

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and when we work out the triple products [cf. Eqs. (19)] we obtain

$$\mathbf{V}' = 2r\mathbf{V} \times \mathbf{R} + (r^2 - R^2)\,\mathbf{V} + 2\mathbf{R}(\mathbf{V} \cdot \mathbf{R}) \tag{51}$$

If $\mathbf{V}$ is written in terms of components parallel and perpendicular to the rotation direction and use is made of Table I, we arrive at the more familiar expressions

$$V_{\parallel}' = V_{\parallel}, \qquad \mathbf{V}_{\perp}' = (\sin\theta)\,\mathbf{V} \times \hat{R} + (\cos\theta)\,\mathbf{V}_{\perp}, \qquad v' = v \tag{52}$$

These constitute the well-known formulas for a space rotation.

In like manner we merely work out the details to demonstrate that the boost transformation

$$\{\mathbf{V}', iv'\} = [i\mathbf{B}, b] \,\square\, \{\mathbf{V}, iv\} \,\square\, [i\mathbf{B}, b] \tag{53}$$

gives the result

$$\begin{aligned} \mathbf{V}' &= \mathbf{V}(b^2 - \mathbf{B}^2) + 2\mathbf{B}(\mathbf{B} \cdot \mathbf{V}) - 2vb\mathbf{B} \\ v' &= v(b^2 + B^2) - 2b\mathbf{V} \cdot \mathbf{B} \end{aligned} \tag{54}$$

which may be written in the more common form

$$V'_{\parallel} = \gamma(\mathbf{V}_{\parallel} - \beta v), \qquad \mathbf{V}_{\perp}' = \mathbf{V}_{\perp}, \qquad v' = \gamma(v - \beta V_{\parallel}) \tag{55}$$

corresponding to a boost in the parallel direction, where use was made of Table I.

Thus we have demonstrated that the transformation law (40) of a four vector is valid for a space rotation and a boost. The validity of this transformation law (40) for more general transformations (30) is easily proven by induction.

6. ELECTROMAGNETIC POTENTIALS AND FIELDS

Now that the relativistic vector approach has been applied to special relativity, we will proceed to apply it to the case of electromagnetism.

The electromagnetic fields **H** and **E** form a second-order relativistic vector F which may be constructed from the gradient four-vector

$$\nabla \equiv \{-\boldsymbol{\nabla}, (i/c)\, \partial/\partial t\} \tag{56}$$

and the electromagnetic potential four-vector A

$$A \equiv \{\mathbf{A}, i\phi\} \tag{57}$$

With the aid of Eq. (14) we write for F

$$F = \nabla \,\square\, A^* = \{-\boldsymbol{\nabla}, (i/c)\, \partial/\partial t\} \,\square\, \{\mathbf{A}, -i\phi\} \tag{58}$$

To carry out the square product operation we must be careful to conform to the order of terms in Eq. (58) by always keeping the operator parts ∇ and $\partial/\partial t$ to the left of the potential terms $\mathbf{A}$ and ϕ. Accordingly we have

$$F = \left[\nabla \times \mathbf{A} + i\left(\nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right), \nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t}\right] = [\mathbf{H} - i\mathbf{E}, 0] \qquad (59)$$

where the electromagnetic potentials have the standard definitions

$$\mathbf{H} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\phi - (1/c)\,\partial\mathbf{A}/\partial t \qquad (60)$$

and the scalar part vanishes if the Lorentz condition is used

$$\nabla \cdot \mathbf{A} + (1/c)\,\partial\phi/\partial t = 0 \qquad (61)$$

Since $(\mathbf{H} - i\mathbf{E}, 0)$ has six components it might be referred to as a six-vector, and we will call it the electromagnetic six-vector. Its form in a new Lorentz frame is obtained from Eq. (43)

$$[\mathbf{H}' - i\mathbf{E}, 0] = [\mathbf{H}_{\parallel} + \gamma(\mathbf{H}_{\perp} + \boldsymbol{\beta} \times \mathbf{E}) - i\mathbf{E}_{\parallel} - i\gamma(\mathbf{E}_{\perp} - \boldsymbol{\beta} \times \mathbf{H}), 0] \quad (62)$$

The real and imaginary parts of the magnitude of F

$$F \cdot F = 2(H^2 - E^2) - 2i\mathbf{H} \cdot \mathbf{E} \qquad (63)$$

provide, respectively, the two invariants $(H^2 - E^2)$ and $\mathbf{H} \cdot \mathbf{E}$ of the electromagnetic fields.

7. MAXWELL'S EQUATION AND THE WAVE EQUATIONS

Maxwell's equations are obtained by forming a third-order relativistic vector though the operation with Δ on the conjugate field six-vector F^* in accordance with Eq. (20),

$$\nabla \,\square\, F^* = \left\{-\nabla, \frac{i}{c}\frac{\partial}{\partial t}\right\} \square\, [\mathbf{H} + i\mathbf{E}, 0] = V + V_p \qquad (64)$$

where use was made of Eq. (21). We identify V as the current charge density four-vector J

$$V = (4\pi/c)J = (4\pi/c)(\mathbf{J}, ic\rho) \qquad (65)$$

and set V_p equal to zero, to obtain

$$\left\{\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + i\left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right), \nabla \cdot \mathbf{H} + i\nabla \cdot \mathbf{E}\right\} = \frac{4\pi}{c}\{\mathbf{J}, ic\rho\} \tag{66}$$

This single expression constitutes a statement of Maxwell's equations,

$$\begin{aligned} \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi\mathbf{J}}{c}, \qquad & \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} &= 0, & \nabla \cdot \mathbf{E} &= 4\pi\rho \end{aligned} \tag{67}$$

as may be seen by equating the real and imaginary, scalar and vector parts of Eq. (66).

The potential wave equations are obtained by forming the following third-order relativistic vector from Eq. (58):

$$\left\{-\nabla, \frac{i}{c}\frac{\partial}{\partial t}\right\} \square \left[\nabla \times \mathbf{A} - i\left(\nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right), 0\right] = \frac{4\pi}{c}\{\mathbf{J}, ic\rho\} \tag{68}$$

The operations may be carried out to give

$$\left\{\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}, i\left(\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right)\right\} = \frac{4\pi}{c}\{\mathbf{J}, ic\rho\} \tag{69}$$

where use was made of the Lorentz condition (61). The real vector part is the wave equation for the vector potential

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{4\pi\mathbf{J}}{c} \tag{70}$$

and the imaginary scalar part is the wave equation for the scalar potential

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 4\pi\rho \tag{71}$$

A similar approach using the gradient operator ∇ provides the wave equation for the electromagnetic fields as a fourth-order relativistic vector (23),

$$\begin{aligned} &\left\{-\nabla, \frac{1}{c}\frac{\partial}{\partial t}\right\} \square \left\{\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - i\left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right), \nabla \cdot \mathbf{H} - i\nabla \cdot \mathbf{E}\right\} \\ &\quad = \left[-\nabla, \frac{i}{c}\frac{\partial}{\partial t} \square \frac{4\pi}{c}\{\mathbf{J}, -ic\rho\}\right] \end{aligned} \tag{72}$$

The square cross operations are easily carried out to give

$$\left[\nabla^2\mathbf{H} - \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} - i\left(\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right), 0\right]$$

$$= \frac{4\pi}{c}\left[-\nabla\times\mathbf{J} - i\left(\nabla\rho + \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t}\right), \nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t}\right] \tag{73}$$

The real vector part of these six-vectors is the wave equation for the magnetic field

$$\nabla^2\mathbf{H} - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = -\frac{4\pi\nabla}{c}\times\mathbf{J} \tag{74}$$

the imaginary vector part is the wave equation for the electric field

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 4\pi\left(\nabla\rho + \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t}\right) \tag{75}$$

and the scalar part is the continuity equation

$$\nabla\cdot\mathbf{J} + \partial\rho/\partial t = 0 \tag{76}$$

Thus the present formalism provides a very compact way to express the equations of electromagnetism.

8. SPACE–TIME INVERSION AND IMPROPER TRANSFORMATIONS

Until now the discussion has been confined to proper transformations which have a magnitude of + 1. Such proper transformations exclude the operations of space inversion and time reversal. The transformation I

$$I = [0, i] \tag{77}$$

which has the reciprocal (0, $-i$) and the magnitude -1, produces a total space–time inversion of an odd-order relativistic vector since the operation $I \square V \square I^{*-i}$ converts both the vector and scalar parts of a four-vector and a pseudo-four-vector to their negatives in accordance with Eqs. (40) and (42), respectively,

$$\begin{aligned}\{-\mathbf{V}, -iv\} &= [0, i] \square \{\mathbf{V}, iv\} \square [0, i]\\ \{-i\mathbf{V}_p, iv_p\} &= [0, i] \square \{i\mathbf{V}_p, -v_p\} \square [0, i]\end{aligned} \tag{78}$$

I differs from other transformations because it cannot be written as the product of two four-vectors. Even-order relativistic vectors undergo total inversion by using Eq. (45b), where T^{-1} equals $[0, -i]$.

A proper transformation T is converted to an improper transformation T_p with a magnitude -1 through the operation

$$T_p = I \square T = T \square I \tag{79}$$

corresponding to

$$[T_p, t_p] = [i\mathbf{T}, it] \tag{80}$$

Using the lower sign of Eq. (37), we obtain $[\mathbf{T}_p, -t_p]$ as the reciprocal of an improper transformation,

$$[\mathbf{T}_p, t_p]^{-1} = [\mathbf{T}_p, -t_p] \tag{81}$$

The successive application of $[0, i]$ or $[0, -i]$ twice produces the self reciprocal transformation $[0, -1] = -U$,

$$I \square I = I^* \square I^* = -U$$

which has no effect on a four-vector

$$\{\mathbf{V}, iv\} = [0, -1] \square \{\mathbf{V}, iv\} \square [0, -1] \tag{82}$$

Using these four transformations $[0, 1]$, $[0, i]$, $[0, -1]$, and $[0, -i]$, we can divide the set or group of general space–time transformations into four branches:

$$\begin{aligned} [\mathbf{T}, t] \square [0, 1] &= [\mathbf{T}, t] && \text{proper branch} \\ [\mathbf{T}, t] \square [0, i] &= [i\mathbf{T}, it] && \text{improper branch} \\ [\mathbf{T}, t] \square [0, -1] &= [-\mathbf{T}, -t] && \text{proper double branch} \\ [\mathbf{T}, t] \square [0, -i] &= [-i\mathbf{T}, -it] && \text{improper double branch} \end{aligned} \tag{83}$$

Each double branch has a one to one correspondence with its counterpart single or ordinary branch, and both have the same effect on the transformation of a four-vector since from Eq. (82), $[\mathbf{T}, t]$ is effectively the same transformation as $[-\mathbf{T}, -t]$.

In the case of boosts the two branches correspond to the following disjoint range of values of the parameter b:

$$\begin{aligned} 1 \leqslant b < \infty && \text{ordinary branch} \\ -\infty < b \leqslant -1 && \text{double branch} \end{aligned} \tag{84}$$

For space rotations the branches are connected, and if we select $-\pi \leqslant \theta \leqslant \pi$ for the range of θ corresponding to the ordinary branch, we obtain

$$\begin{array}{lll} 0 \leqslant r \leqslant 1 & \text{ordinary branch} & -\pi \leqslant \theta \leqslant \pi \\ -1 \leqslant r \leqslant 0 & \text{double branch} & -2\pi \leqslant \theta \leqslant -\pi, \quad \pi \leqslant \theta \leqslant 2\pi \end{array} \tag{85}$$

The existence of the two space rotation double branches is associated with the well-known double-valuedness of spinor matrices.

9. PARITY AND TIME REVERSAL

The four-vector $\{\mathbf{r}, ict\}$ is reversed in time by the operation of complex conjugation, and parity P may be considered as a combination of time reversal T and total inversion I. Therefore we have

$$P\{\mathbf{V}, iv\} = [0, i] \square \{\mathbf{V}, iv\}^* \square [0, i] = \{-\mathbf{V}, iv\} \tag{86}$$

$$T\{\mathbf{V}, iv\} = \{\mathbf{V}, iv\}^* = \{\mathbf{V}, -iv\} \tag{87}$$

$$I\{\mathbf{V}, iv\} = [0, i] \square \{\mathbf{V}, iv\} \square [0, i] = \{-\mathbf{V}, -iv\} \tag{88}$$

These three operations commute with each other; e.g., $P = IT = TI$. The corresponding expressions for a pseudo-four-vector V_p are

$$P\{i\mathbf{V}_p, -v_p\} = (i\mathbf{V}_p, v_p\} \tag{89}$$

$$T\{i\mathbf{V}_p, -v_p\} = \{-i\mathbf{V}_p, -v_p\} \tag{90}$$

$$I\{i\mathbf{V}_p, -v_p\} = \{-i\mathbf{V}_p, v_p\} \tag{91}$$

We see from the sign changes of Eqs. (86) and (89) that $\mathbf{V}$ is a vector, $\mathbf{V}_p$ is a pseudovector, v is a scalar, and v_p is a pseudoscalar. Hence the notation of pseudo-four-vector for V_p.

10. MAGNETIC MONOPOLES

Maxwell's equations were derived above by equating $\nabla \square F^*$ to the current-charge density four-vector J. The case of magnetic monopoles may be treated by writing for Eq. (64)

$$\left\{-\nabla, \frac{i}{c}\frac{\partial}{\partial t}\right\} \square [\mathbf{H} + i\mathbf{E}, 0] = \frac{4\pi}{c}\{\mathbf{J}, ic\rho\} - \frac{4\pi}{c}\{i\mathbf{J}_m, -c\rho_m\} \tag{92}$$

where $\mathbf{J}_m$ and ρ_m are the magnetic current and magnetic charge density, respectively, corresponding to V_p of Eq. (22), and the signs on $\mathbf{J}_m$ and ρ_m are selected to agree with Jackson.[8] This expression gives for the space part

$$\left(\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) + i\left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial H}{\partial t}\right) = \frac{4\pi}{c}(\mathbf{J} - i\mathbf{J}_m) \qquad (93)$$

and for the time part

$$\nabla \cdot \mathbf{H} + i\,\nabla \cdot \mathbf{E} = 4\pi(\rho_m + i\rho) \qquad (94)$$

which symmetrizes the real and imaginary parts, as expected for a magnetic monopole formulation. The form of Eq. (92) correlates with the fact that $\mathbf{J}$ and $c\rho$ constitute a polar vector–scalar pair, and $\mathbf{J}_m$ and $c\rho_m$ constitute a pseudovector–pseudoscalar pair. We should note from the discussion after Eq. (46) that the relativistically invariant quantities of the J, J_m pair of first-order vectors are $J^2 - c^2\rho^2$, $-J_m{}^2 + c^2\rho_m{}^2$, and $\mathbf{J} \cdot \mathbf{J}_m - c^2\rho\rho_m$.

11. CLIFFORD ALGEBRAS

The formulation presented in this and the two previous articles is closely related to Clifford algebras, and indeed it may be considered as a particular application of Clifford algebras. In this section we will comment upon the relationship between the present approach and the following Clifford algebras[5,6]:

1. The real quaternion algebra C_2 operates on a linear vector space of dimension $2^2 = 4$ with the orthonormal basis vectors given by $\hat{i}, \hat{j}$ (where $\hat{k} = \hat{i} \times \hat{j}$). The vector products (3) and (4) are related to the usual quaternion product

$$c + i\mathbf{C} = (b + i\mathbf{B})(a + i\mathbf{A}) = ab - \mathbf{AB} + i(a\mathbf{B} + b\mathbf{A}) \qquad (95)$$

through expression (1.30) of Ref. 5,

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + i\mathbf{A} \times \mathbf{B} \qquad (96)$$

The operation $\mathbf{AB}$ is not defined in the present formalism.

The first article in this series dealt with the rotations of three-dimensional vectors in Cartesian space. The rotation vectors $(\mathbf{R}, r)$ presented there were real unit quaternions or quaternions of unit length.

2. The Pauli algebra or algebra of complex quaternions C_3 is associated with a linear vector space of dimension $2^3 = 8$ with the orthonormal basis vectors given by the Pauli spin matrices σ_1, σ_2, and σ_3.

3. The Dirac algebra C_4 is determined by a linear vector space of dimension $2^4 = 16$ with the orthonormal basis vectors given by the Dirac matrices γ_0, γ_1, γ_2, and γ_3.

The elements of the Dirac algebra C_4 are of five types: one-component scalars S_D, four-component vectors V_D, six-component trivectors (tensors) T_D, four-component pseudovectors (also called axial vectors or bivectors) B_D, and one-component pseudoscalars P_D. The elements of the Pauli algebra C_3 are of four types: one-coponent scalars S_p, three-component vectors V_p, three-component pseudovectors B_p, and one-component pseudoscalars P_p. Each of these terms plus the terms of the algebras of real numbers (C_0), complex numbers (C_1), and also C_5 may be arranged in the array shown in Table III. In this array we indicate for each term its dimensionality and its total parity (evenness of oddness) under total inversion. We see from the array that all scalars are even and of dimension unity, all vectors are odd and of dimension n, all terms in a row alternate in total parity from plus to minus, and all C_n dimensionalities are binomial coefficients (e.g., 1-4-6-4-1 for C_4). For simplicity the subscripts are omitted from the zeroth-order, first-order, second-order, and fifth-order terms.

For each algebra C_n there are two subspaces formed from C_{n-1} by grouping each term of C_{n-1} in pairs or singly with a term of C_n that appears in the array to the right or left directly below. This grouping is done in a way that preserves the dimensionality.

The odd-order relativistic vectors used in this work correspond to the odd Pauli subspace with the following decomposition:

$$V_D = V_p + iS_p, \qquad B_D = iB_p + P_p \tag{97}$$

The first-order relativistic vector V_D alone is an incomplete part of this subspace. Since Eq. (97) constitutes the complete subspace, odd higher order

Table III.

C_n	Terms of Algebra						Dimension n
C_0	$S_{(1,+)}$						1
C_1	$S_{(1,+)}$	$P_{(1,-)}$					2
C_2	$S_{(1,+)}$	$V_{(2,-)}$	$P_{(1,+)}$				4
C_3	$S_{p(1,+)}$	$V_{p(3,-)}$	$B_{p(3,+)}$	$P_{p(1,-)}$			8
C_4	$S_{D(1,+)}$	$V_{D(4,-)}$	$T_{D(6,+)}$	$B_{D(4,-)}$	$P_{D(1,+)}$		16
C_5	$S_{(1,+)}$	$V_{(5,-)}$	$T_{(10,+)}$	$T'_{(10,-)}$	$B_{(5,+)}$	$P_{(1,-)}$	32

relativistic vectors do not generate any additional types of elements. Therefore, Eq. (21) is equivalent to the expression

$$Q = V_D + B_D \tag{98}$$

when written in the notation of Eq. (97).

The even-order relativistic vectors correspond to the following even Pauli subspace decomposition:

$$S_D = S_p\,, \qquad T_D = iV_p + B_p\,, \qquad P_D = iP_p \tag{99}$$

Second-order relativistic vectors lack P_D and fourth-order ones contain the complete subspace (99). Equation (25) expressed in this notation has the form

$$H = T_D + S_D + P_D \tag{100}$$

These odd and even Pauli subspaces, given by Eqs. (97) and (99), respectively, exhaust the Dirac algebra. The even Pauli subspace is a subalgebra because it is closed under the operation $\square$, while the odd Pauli subspace is not closed under this operation and hence it is not a subalgebra.

In forming subspaces the i is inserted in front of each term of the subspace that differs in total parity from the higher order algebra term associated with it. For example, V_D is odd and hence even S_p has an i and odd V_p has no i in Eq. (97). In like manner, the even Dirac term T_D of Eq. (99) contains the Pauli terms iV_p and B_p.

The transformations used in this work mix the elements of the Pauli subalgebra which are associated together with each Dirac algebra term, but they do not allow mixing between Dirac algebra terms. Thus, transormations of Q [Eq. (98)] in the odd subspaces (97) mix together elements of V_p and S_p, and mix those of B_p and P_p, but no other mixing occurs. In the even subalgebra (99) elements of iV_p and B_p mix together, while S_p and iP_p remain invariant under transformations of H [Eq. (100)]. In other words, the special and general Lorentz transformations (29) and (30), respectively, leave these odd and even subspaces separately invariant through Eqs. (45a) and (45b). This explains why in the electromagnetic field six-vector E and B mix, but the zero time part ($S_p = 0$, $P_p = 0$) remains zero.

The transformations themselves correspond to the even Pauli subalgebra. Thus the scalar part b in Eq. (48) remains invariant. More specifically, the transformations are even-order elements of the Pauli algebra with unit magnitude, and as a result they preserve the magnitude of the relativistic vectors which they transform.

12. DISCUSSION

In this article we have shown that the use of a linear vector product formalism permits a four-vector and its conjugate to be employed as primitive vectors for the formation of quantities called second-, third- and fourth-order relativistic vectors which comprise the space rotations, Lorentz transformations, and antisymmetric tensors of special relativity. The use of these relativistic vectors provides a very compact way to express the equations of electromagnetism. Using this method, all four of Maxwell's equations reduce to one equation (66), the wave equations for the scalar and vector potentials appear together in one expression (69), and a third equation (73) combines the wave equation for the magnetic field, the wave equation for the electric field, and the continuity equation for the electric current and the charge density. The compactness of these expressions serves to emphasize the interrelatedness of the various vectors and scalars in electromagnetism. Others[6,9,11] have made use of the form $\mathbf{E} - i\mathbf{H}$.

The Lorentz group consists of four disjoint parts, namely the ordinary or orthochronous part, and the three parts generated respectively by the space inversion, the time reversal, and the space–time inversion operations.[7] Our transformations only generate two of these parts, namely the proper orthochronous part and the space–time inverted part. Each of these, however, appears in both ordinary and double-valued branches. The operations of space inversion, time reversal, and total inversion are not derivable from primitive four-vectors. Instead, time reversal is produced by complex conjugation, and space inversion is obtained by a combined total inversion and complex conjugation operation.

The general transformation $[\mathbf{T}, t]$ has eight parameters since the scalar t and each vector component of $\mathbf{T}$ are in general complex numbers. The normalization condition is equivalent to two conditions

$$\mathrm{Re}(\mathbf{T} \cdot \mathbf{T} + t^2) = 1, \qquad \mathrm{Im}(\mathbf{T} \cdot \mathbf{T} + t^2) = 0 \tag{101}$$

which leaves six independent parameters, the same as the number in the Lorentz group. A four-vector, with a real vector part, an imaginary scalar part, and an arbitrary real magnitude, has four independent parameters, and a second-order relativistic vector, with a complex vector part, a real scalar part, and an arbitrary complex magnitude, has seven independent parameters. The special case with a zero scalar part, exemplified by the electromagnetic field six-vector, has six parameters. Third- and fourth-order relativistic vectors have in general eight independent parameters.

In this article the emphasis was on the methods of transforming relativistic vectors and on applications to electromagnetism. The results, however, are quite general and may be applied to other aspects of special relativity, such as to the basic equations of mechanics. In relativistic mechanics force–power and momentum–energy form four-vectors and angular momentum and torque form six-vectors.

APPENDIX

Equation (18) of *I* for the triple product of three successive rotations

$$[\mathbf{R}, r] = [\mathbf{R}_3, r] \square [\mathbf{R}_2, r] \square [\mathbf{R}_1, r]$$

should read as follows:

$$\begin{aligned} \mathbf{R} = {} & r_1(\mathbf{R}_2 \times \mathbf{R}_3) - r_2(\mathbf{R}_3 \times \mathbf{R}_1) + r_3(\mathbf{R}_1 \times \mathbf{R}_2) - \mathbf{R}_1(\mathbf{R}_2 \cdot \mathbf{R}_3) \\ & + \mathbf{R}_2(\mathbf{R}_3 \cdot \mathbf{R}_1) - \mathbf{R}_3(\mathbf{R}_1 \cdot \mathbf{R}_2) + r_2 r_3 \mathbf{R}_1 + r_3 r_1 \mathbf{R}_2 + r_1 r_2 \mathbf{R}_3 \\ r = {} & r_1 r_2 r_3 - r_1(\mathbf{R}_2 \cdot \mathbf{R}_3) - r_2(\mathbf{R}_3 \cdot \mathbf{R}_1) - r_3(\mathbf{R}_1 \cdot \mathbf{R}_2) + \mathbf{R}_2 \cdot (\mathbf{R}_3 \times \mathbf{R}_1) \end{aligned}$$

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REFERENCES

1. Y. Aharonov, H. A. Farach, and C. P. Poole, Jr., *Am. J. Phys.* **45**, 451 (1977); referred to as I.
2. H. A. Farach, Y. Aharonov, and C. P. Poole, Jr., *Am. J. Phys.* **47**, 247 (1979); referred to as II.
3. W. K. Clifford, *Am. J. Math.* **1**, 350 (1878).
4. B. L. van der Waerden, *Algebra* (Springer Verlag, Berlin, 1967).
5. D. Hestenes, *Space–Time Algebra* (Gordon and Breach, New York, 1966).
6. D. Hestenes, *Am. J. Phys.* **39**, 1013 (1971).
7. S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1962), Chapter 2.
8. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
9. A. Sommerfeld, *Electrodynamics* (Academic Press, New York, 1952).
10. A. O. Barut and S. Malin, *Found. Phys.* **5**, 375 (1975).
11. A. A. Frost, *Found. Phys.* **5**, 619 (1975).