# A REPLY TO "GAUGE INVARIANCE AND EXPERIMENTAL PROCESSES"

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We reply to the charges by K.H. Yang in his paper "Gauge invariance and experimental processes".

Recently in a paper entitled "Gauge invariance and experimental processes: an experimentalist's point of view", K.H. Yang [1] again advocates that in the calculation of quantum-mechanical transition amplitudes for a nonrelativistic spinless particle of mass $m$ and charge $e$ subject to an electrostatic field $\boldsymbol{E}_0(\boldsymbol{r}) = -\nabla V_0(\boldsymbol{r})$ and an electromagnetic radiation field $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ such that $(A, \Phi)$ is the 4-potential that gives rise to $\boldsymbol{E}$ and $\boldsymbol{B}$, there exists a preferred operator in the form

$$H_B = (\boldsymbol{p} - eA)^2/2m + eV_0 , \qquad (1)$$

the time rate of change of which has a classical analog which Yang identified as the power. The hamiltonian is

$$H = (\boldsymbol{p} - eA)^2/2m + eV_0 + e\Phi , \qquad (2)$$

with a wave function $\psi(\boldsymbol{r}, t)$. Yang states that if $\{\psi_j^B(\boldsymbol{r}, t)\}$ is the orthonormal and complete set of eigenfunctions of $H_B(t)$ with corresponding eigenvalues $\{E_j^B(t)\}$, , then the probability amplitude relevant to experimental measurement is given by

$$a_j(t) \equiv \langle \psi_j^B(\boldsymbol{r}, t) | \psi(\boldsymbol{r}, t)\rangle . \qquad (3)$$

In the same paper, Yang also claims that the present authors [2] "have erred" in the definition of gauge invariance and goes on further to state "the fact that Aharonov and Au have 'explained all the facts' about how to measure the 'unperturbed' hamiltonian illustrates a lack of knowledge of the actual experimental processes in what quantities are measured in an electromagnetic field–matter interaction". The purpose of the present note is to point out that Yang has missed the points in our paper and further misinterpreted our paper for the readers on three issues.

Firstly, never anywhere in ref. [2] have we advocated the measurements of the unperturbed hamiltonian as Yang states in his paper. In fact, we never discussed which measurements are to be preferred over others. Secondly, we stated that any experimentally measurable quantities can be calculated in any gauge. Thirdly, as we painfully tried to emphasize in our paper, each scalar product of two state vectors in Hilbert space is a probability amplitude, the experimental relevance of which is specified once the gauge under which the state wave function is calculated is specified. This scalar product may or may not be gauge invariant. Nevertheless, it is always measurable. It is this third point that we believe Yang has missed in our paper.

To clarify more about the second issue, let us first examine what physical experimental measurement the scalar product $a_j \equiv \langle \psi_j^B | \psi \rangle$ corresponds to. As we state in our earlier paper, physically measurable quantities are functions of velocity and position, which are necessarily gauge invariant. Since $\psi$ is the solution to the Schrödinger equation whose hamiltonian is given by eq. (2), the velocity operator corresponding to this particular gauge is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \partial H/\partial \boldsymbol{p} = (\boldsymbol{p} - eA)/m . \qquad (4)$$

Hence the operator $H_B$ in the same gauge corresponds to

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$$H_B = \tfrac{1}{2}mv^2 + eV_0 , \tag{5}$$

which is identifiable as a function of the gauge-invariant quantities $\mathbf{v}$ and $\boldsymbol{r}$. Thus the amplitude $a_j \equiv \langle \psi_j^B | \psi \rangle$ is the probability amplitude to find the particle at time $t$ to have an eigenvalue $E_j$ for the operator $H_B$ in eq. (5). This is definitely something that can be measured and, in fact, may well be what Yang's experiment is all set up to measure. However, as we showed in ref. [2], the same probability amplitude $a_j$ can be calculated in any gauge. Consider a c-number gauge transformation that takes the 4-potential $(A, \Phi)$ to $(A', \Phi')$ such that

$$A' = A + \nabla\chi , \tag{6a}$$

and

$$\Phi' = \Phi - \partial\chi/\partial t . \tag{6b}$$

Then the hamiltonian in eq. (2) is transformed to

$$H' = (p - eA')^2/2m + eV_0 + E\Phi' , \tag{7}$$

which is obtained from $H$ via the transformation

$$H' = UHU^+ \mathrm{i}\dot{U}^+ \mathrm{i}\dot{U}U^+ , \tag{8}$$

where

$$U \equiv \exp(\mathrm{i}e\chi) . \tag{9}$$

Let $\psi'(t)$ be the wave function that is the solution to the Schrödinger equation with the hamiltonian $H'$ in eq. (7). Then

$$\psi' = U\psi . \tag{10}$$

If one is interested in measuring the same gauge-invariant quantity, as expressed in eq. (5), but in a different gauge:

$$f(\mathbf{v}, \boldsymbol{r}) = \tfrac{1}{2}mv^2 + eV_0 , \tag{11}$$

the relevant transition amplitude is

$$a_j' = \langle \phi_j' | \psi' \rangle , \tag{12}$$

where $|\phi_j'\rangle$ satisfies

$$[(p - eA')^2/2m + eV_0] |\phi_j'\rangle = E_j |\phi_j'\rangle , \tag{13}$$

since in the gauge $H'$ is written,

$$\mathbf{v} = \partial H'/\partial p = (p - eA')/m . \tag{14}$$

Thus $a_j'$ is the probability amplitude to find the system with $f(\mathbf{v}, \boldsymbol{r})$ in eq. (11), which can be identified as the free energy, to have a value $E_j$. On the other hand

$$|\phi_j'\rangle = U |\psi_j^B\rangle , \tag{15}$$

since

$$UH_B U = (p - eA')^2/2m + eV_0 \equiv H_B' . \tag{16}$$

Thus

$$a_j' = \langle \phi_j' | \psi' \rangle = \langle \psi_j^B | U^\dagger U | \psi \rangle = \langle \psi_j^B | \psi \rangle = a_j , \tag{17}$$

illustrating once again that there is no preferred gauge in the calculation of physical transition amplitude.

To reemphasize the third point, we consider a complete orthogonal set of eigenfunctions $\{\phi_j\}$ of a certain operator $O$ with the corresponding set of eigenvalues $\beta_j$. Our point is that the scalar product

$$b_j \equiv \langle \phi_j | \psi' \rangle \tag{18}$$

has definite experimental implications once the gauge under which $\psi'$ is evaluated is specified. The first thing one has to do is to identify the physical gauge-invariant quantities that the operator $O$ corresponds to in the gauge that $H'$ is written. $O$ is a function of $\boldsymbol{r}$ and the differential operator $\nabla$ which is identified as the canonical momentum operator $\boldsymbol{p}$. Hence, we identify

$$O = O(\boldsymbol{p}, \boldsymbol{r}) . \tag{19}$$

Next, we express $\boldsymbol{p}$ in terms of the gauge-invariant quantities $\mathbf{v}$ and $\boldsymbol{r}$ via eq. (14):

$$p = m\mathbf{v} + eA'(\boldsymbol{r}, t) . \tag{20}$$

Hence

$$O = O(m\mathbf{v} + eA', \boldsymbol{r}) \equiv f_\chi(\mathbf{v}, \boldsymbol{r}) , \tag{21}$$

which is a known function of the gauge-invariant quantities $\mathbf{v}$ and $\boldsymbol{r}$. One can certainly set up an experiment to measure this function $f_\chi(\mathbf{v}, \boldsymbol{r})$. This quantity may not be what Yang's photon counters are set up to measure. However, in principle, it is possible to measure this function $f_\chi(\mathbf{v}, \boldsymbol{r})$. Hence, $b_j \equiv \langle \phi_j | \psi \rangle$ is the probability amplitude for an experimental measurement to find the particle to have a value $\beta_j$ for the quantity $f_\chi(\mathbf{v}, \boldsymbol{r})$. In particular, $O(\boldsymbol{p}, \boldsymbol{r})$ may happen to be $p^2/2m + eV_0$, but, of course, not limited to it. For example, $O$ could be $H_B$, or $H_B'$. If $O$ is $H_B'$, $b_j$ is indeed gauge invariant. However, the amplitude $b_j$ corresponds to a definite experimental measurement whatever the operator $O$ is, once the gauge under which the state wave function $\psi'$ is calculated is specified.

We wish to emphasize that we have added nothing new in this paper – it has all been said in ref. [2]. But Yang has presented the impression in ref. [1] that we are advocating the measurement of $\langle \phi_j | \psi \rangle$ where $\{\phi_j\}$ is the set of eigenstates for the operator $p^2/2m + eV_0$. We only state in ref. [2] that such a measurement is possible and is different from the measurement which the scalar product $a_j \equiv \langle \psi_j^{\mathrm{B}} | \psi \rangle$ corresponds to.

Lastly, we would like to point out that shortly after Yang's paper [1] was published, Feuchtwang et al. [3] confirmed the results of our earlier paper [2].

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*References*

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