Proc. Int. Symp. Foundations of
Quantum Mechanics, Tokyo, 1983, pp. 10–19

Non-Local Phenomena and the Aharonov-Bohm Effect

Yakir AHARONOV

Department of Physics and Asronomy, Tel Aviv University, Tel Aviv, Israel
and
Department of Physics, University of South Carolina, South Carolina 29208 USA

Non-local phenomena in both classical and quantum theories is discussed, and it it is shown that quantum non-local phenomena have unique dynamical properties not shared by the classical phenomena. A short review of attempts to account for the A-B effect in a local approach is given, followed by a discussion that shows that these attempts are unsuccessful. A new type of non-local phenomena is introduced, and a number of new examples for both types of non-local phenomena are discussed.

§1. Introduction

In recent years it became clear that non-local phenomena and topological effects play an important role in many areas of physics. In this talk we will review the A-B effect, which is common to all gauge theories, and provides a particularly clear example of non-local phenomena. We will show that effects analogous to the A-B effect exist in classical theories as well, but that in quantum theories they acquire dynamical significance which has no classical analogue. We will discuss these dynamical aspects and show that they provide a general characterization of non-local phenomena. Finally we will show how this approach can guide us in finding other families of non-local effects which have no classical analogue.

Let us first review the A-B effect.[1] Imagine a thin solenoid confining a magnetic flux Φ surrounded by an impenetrable barrier (Fig. 1). The region outside the barrier is free of fields and therefore all local experiments confined to this region will be insensitive to the flux. The region seems (locally, at any rate) empty.

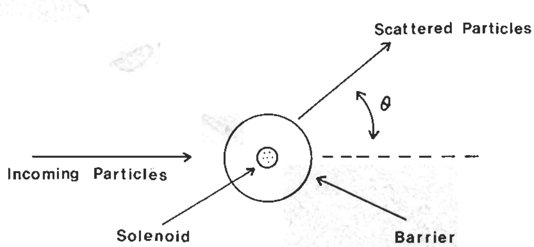

Fig. 1. Scattering from a solenoid surrounded by a barrier.

Indeed, a classical charged particle moving in this region will behave as if nothing whatever is there. On the other hand, the wave-function of a quantum-mechanical charged particle will, in these circumstances, include interference terms which depend on the flux, Φ modulo Φ_0 (where Φ_0 is a unit of flux ch/e called fluxon). Thus the behavior of charged, quantum-mechanical particles is sensitive to the non-vanishing line integral of the vector potential around the barrier. This is an example of non-local phenomena, since the effect of the vector potential in any local region is not gauge invariant and therefore not measurable. We may speak, if we choose to, about a non-local effect of the enclosed flux on the charged particle, since the final effect depends only on the inaccessible flux.

A further insight into the nature of this non-local effect is provided by the scattering of the particle from the barrier surrounding the flux. The scattering cross-section of the particle depends both on R, the radius of the barrier, and on Φ, the flux enclosed therein. If the flux is quantized ($\Phi = n\Phi_0$), the scattering will vanish as $R \to 0$. For non-quantized flux values the scattering does not vanish in that limit, but, rather, is proportional to the wavelength λ of the scattered particle [more particularly: $\sigma \sim \lambda \sin(\pi\Phi/\Phi_0)$]. This periodicity of the scattering cross section as a function of the external "disturbance" (*i.e.*, the external flux) is characteristic of non-local phenomena in general and offers a clue to the dynamical view of such phenomena to be developed below. A second clue is connected with the vanishing of the cross-section in the classical limit.

Before continuing it is perhaps worth disposing of attempts to explain the effect via local means. Certain authors have claimed that under ideal conditions, when the particle is completely confined to the field-free region, the effect will disappear.

That this is not so can be shown as follows: First, the solution of the Schrödinger equation for this ideal case predicts an effect, provided that we use single-valued wave functions. That the wavefunctions *must* be single-valued, even in this ideal case, can be seen from the following argument. Suppose that initially the flux in the solenoid is zero, and is switched on later. The time-dependent Hamiltonian appropriate to these circumstances is manifestly single-valued; the initial state, too, is single-valued, and therefore the final state must be single-valued as well.

As another argument,[2] consider a set-up in which a charged particle q_1 is prepared in a coherent superposition of two states confined to the interior of two perfectly conducting spheres (Fig. 2). A second charge q_2 is placed in the region between the two spheres. By observing the momentum transfer to q_2, we can tell in which of the two spheres q_1 is located, and thus destroy the coherency of the packets. Obviously this can be done even in the ideal case when the state of q_1 vanishes exactly outside the spheres. Thus the relative phase of the state describing q_1 must be sensitive to the fields produced by q_2 even when q_1 is entirely confined to the field free region.

Another attempt to account for the A-B effect locally is based upon the so-called hydrodynamical form of Schrödinger equations.[3] To obtain the hydrodynamical equations, we replace Ψ by ρ and $\boldsymbol{v}$, which are defined as follows:

$$\Psi = Re^{i\Phi} \tag{1}$$

$$\rho = R^2 \tag{2}$$

$$\boldsymbol{v} = \nabla\Phi - \frac{e}{c}\boldsymbol{A}. \tag{3}$$

The equations for ρ and $\boldsymbol{v}$ are:

$$\partial\rho/\partial t + \operatorname{div}\rho\boldsymbol{v} = 0. \tag{4}$$

$$\partial\boldsymbol{v}/\partial t + (\nabla\cdot\boldsymbol{v})\boldsymbol{v} = e\boldsymbol{E} - \nabla(\Delta^2 R/2R) \tag{5}$$

$$\nabla\times\boldsymbol{v} = \frac{e}{c}\nabla\times\boldsymbol{B} \tag{6}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the external electric and magnetic fields respectively. (We have assumed for simplicity $m=1$ and taken $\hbar=1$).

Since these equations are manifestly local and gauge invariant, we may assume that they account locally for the A-B effect. The situation, however, is not so simple. Consider an initial condition where we have finite quantities of fluid in two non-overlapping regions, while in the region separating them there is only infinitesimal quantity (Fig. 3). Let ρ_1, $\boldsymbol{v}_1$ and ρ_2, $\boldsymbol{v}_2$ represent the properties of the fluid in the two non-overlapping regions, and ρ_{in}, $\boldsymbol{v}_{in}$ those of the fluid which occupies the region in between. An electric field is then switched on and off again in the intermediate region and changes $\boldsymbol{v}_{in}$ in eq. (5). Later, when ρ_1 and ρ_2 are allowed to meet, the resulting distribution will depend on $\int_1^2 \boldsymbol{v}_{in}\cdot d\boldsymbol{l} \bmod 2\pi$ (since this integral represents the relative phase in the Schrödinger representation). In the hydrodynamical approach we have to assume that the resulting change in the final ρ is due to the non-linear interaction between ρ_{in} and ρ_1 and ρ_2. This interpretation is unacceptable because the final effect on ρ is independent of the size of ρ_{in}.

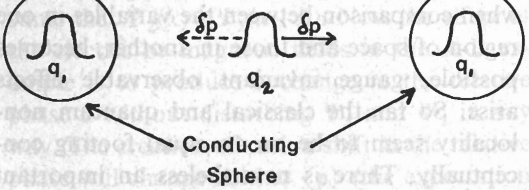

Fig. 2. Interaction between a shielded charge and an external charge.

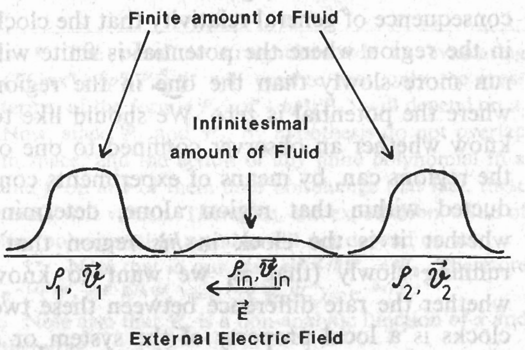

Fig. 3. A-B effect in the hydrodynamical picture.

Thus, even if $\rho_{in} \to 0$, *i.e.* there is initially no fluid at all in the intermediate region, the resulting interaction must nevertheless occur. We are therefore led to the conclusion that $\int_1^2 \boldsymbol{v} \cdot d\boldsymbol{l}$ mod 2π is a non-local property of ρ_1 and ρ_2 rather than a sum of local properties of the fluid in the intermediate region. We have here a very important lesson. Non-linear equations, though local in appearance, may nevertheless conceal non-local effects.

§2. Non-Local Phenomena of the First Type

One hopes that the arguments in the preceding section have convinced the reader that the A-B effect is indeed non-local. I would like now to outline some considerations touching upon the idea of non-locality in general.

Let us, first of all, say quite generally what we mean by a "non-local" property of a physical system. Suppose, we have a system which occupies two separate regions of space (the system might consist, for example, of two objects, one in each region; or, if it is a quantum system, it may consist of a single object whose wave-function is non-zero in these regions, but zero elsewhere). The essential difference between local and non-local properties of the system is that in the former case all possible information can be obtained by independent measurements made in the two regions, while in the latter case this is not true.

Let's start with a classical example. Suppose that in one region there exists a constant (constant in space, that is, but time dependent) gravitational potential, and that in the other the potential is zero,[4)] and that our system consists of two clocks, one in each of the regions. Despite the fact that there is no gravitational force in either region, it is a well-known consequence of general relativity that the clock in the region where the potential is finite will run more slowly than the one in the region where the potential is zero. We should like to know whether an observer confined to one of the regions can, by means of experiments conducted within that region alone, determine whether it is the clock in *his* region that's running slowly (that is, we want to know whether the rate difference between these two clocks is a local property of the system or a non-local one). Since the effect of the gravitational potential is universal, that is, the potential will affect any experimental apparatus of which such an observer makes use in precisely the same way as it does the clock itself, no local determination will be possible. Let the observer compare the clock in question with any "standard" clock in his own region; that latter clock will be slowed by the gravitational potential (or not slowed, if the potential is absent) in precisely the same way as the former. Such a standard clock, then, will necessarily fail to detect any slowing of the original clock relative to itself. Thus the effect of the potential here is a purely non-local one, which can be ascertained only later when a comparison of the times on the two clocks becomes possible.

The quantum-mechanical situation has much in common with this. Suppose that we prepare a charged particle in the following superposition of two localized wave packets,

$$\Psi_+ = \Psi_1 + \Psi_2 \tag{7}$$

where Ψ_1, Ψ_2 are small packets centered at x_1 and x_2. By means of the application of localized electric potentials in the vicinities of x_1 or x_2 (potentials, that is, which are constant in space through the neighborhood of x_1 or x_2 and thus will produce *no* forces on the particle), the phase difference between the the two components can manifestly be altered so as to produce, for example,

$$\Psi_- = \Psi_1 - \Psi_2. \tag{8}$$

But Ψ_- can't be distinguished in any of its *local* properties from Ψ_+. It differs from Ψ_+ only in certain of its observable *non*-local properties.

We thus see that both classical and quantum gauge theories lead to non-local phenomena. In both theories the local description includes non-gauge invariant quantities, *i.e.*, potentials, phases, time, *etc*. These gauge dependent quantities cannot be observed locally. Only later, when comparison between the variables in one region of space and those in another, becomes possible, gauge invariant observable effects arise. So far the classical and quantum non-locality seem to be on an equal footing conceptually. There is nevertheless an important distinction between the two, as we proceed to show in the following section.

§3. Dynamical Considerations

In order to understand the dynamical issues involved in our problem it is best to switch from the Schrödinger picture (which we have been using thus far) to the Heisenberg picture. Consider again the ideal case where an external electric field $\boldsymbol{E}$ affects the behavior of a charged particle which is confined to the field-free region. In the Heisenberg picture it would seem that such an electric field can have no effect on the particle because the Heisenberg equations of motion for $\boldsymbol{x}$ and $\boldsymbol{p}$ are identical, in this case, to the classical ones, *i.e.*

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\frac{\boldsymbol{p}}{m}$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=e\boldsymbol{E} \tag{9}$$

Since $\boldsymbol{E}$ vanishes where the particle is located, these equations of motion seem not to depend on $\boldsymbol{E}$ at all. However, this first impression is misleading.[5)] While for classical theory the vanishing of $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$ implies the vanishing of $\mathrm{d}f(\boldsymbol{p})/\mathrm{d}t$, where $f(\boldsymbol{p})$ is any arbitrary function of $\boldsymbol{p}$, this is not necessarily true quantum mechanically. Indeed, consider the function

$$f(\boldsymbol{p})=\exp\,(i\boldsymbol{p}\cdot\boldsymbol{L}/\hbar) \tag{10}$$

(*i.e.* the translation operator). Then

$$\frac{\mathrm{d}f}{\mathrm{d}t}=[f(\boldsymbol{p}_1),\,H]=\left(\int_{\boldsymbol{x}}^{\boldsymbol{x}+\boldsymbol{L}} \boldsymbol{E}\cdot d\boldsymbol{l}\right)\cdot f \tag{11}$$

where $\boldsymbol{x}$ is the position of the particle. The Heisenberg equations of motion are then manifestly non-local for $f(\boldsymbol{p})$ of the type considered above.

In order to relate this result to the Schrödinger picture consider the state:

$$\Psi_\alpha=\Psi_1+e^{i\alpha}\,\Psi_2 \tag{12}$$

where again Ψ_1 and Ψ_2 occupy non-overlapping regions (Fig. 3). Let us also assume that $\Psi_2(x)=\Psi_1(x+L)$. Now the effect of the electric or magnetic fields in the A-B example is to produce a change in the relative phase α without disturbing Ψ_1 and Ψ_2 in any way. The creation of such a shift in the relative phase will change neither $\langle \boldsymbol{p}\rangle$ (the expectation value of the momentum of the particle) nor $\langle \boldsymbol{x}\rangle$ nor the expectation value of any polynomial in $\boldsymbol{x}$ and $\boldsymbol{p}$.* The expectation value of the translation operator $e^{i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar}$, though, depends on α since for this case

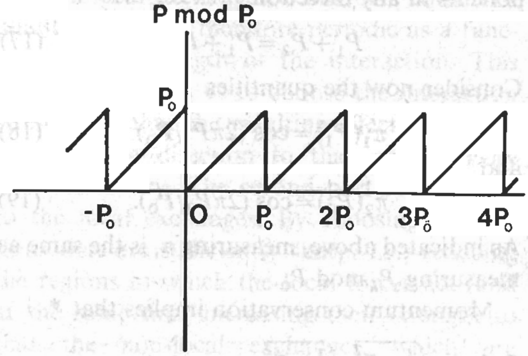

Fig. 4. Modulo Momentum.

$$\langle e^{i\boldsymbol{p}\cdot\boldsymbol{L}/\hbar}\rangle=e^{-i\alpha}/2^{**} \tag{13}$$

Let us proceed by noticing that the eigenstates of the translation operator $\exp\,(i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar)$ are also eigenstates of the modular momentum $\boldsymbol{P}$ modulo $\boldsymbol{P}_0$ (Fig. 4) defined by:

$$\boldsymbol{P}\ \mathrm{mod}\ \boldsymbol{P}_0=\boldsymbol{P}-\boldsymbol{n}\boldsymbol{P}_0. \tag{14}$$

Here $\boldsymbol{n}$ is an integer vector satisfying

$$0\leqq \boldsymbol{P}-\boldsymbol{n}\boldsymbol{P}_0\leqq \boldsymbol{P}_0, \tag{15}$$

componentwise and

$$P_0=h/L. \tag{16}$$

We see then that the non-local effect of the fields is to produce a shift in the modular momentum of the charged particle while leaving the expectation values of moments of its momentum unaltered.

We next show that modular variables satisfy conservation laws of their own. Consider a collision between two systems, 1 and 2. Momentum conservation implies that for com-

---

* The proof is straightforward: In evaluating $\langle\Psi_\alpha|ax^n+bp^n|\Psi_\alpha\rangle$ it will emerge that only the cross terms, of the form $\langle\Psi_1|ax^n+bp^n|\Psi_2\rangle$ will depend on α. Now, since Ψ_1 and Ψ_2, by hypothesis do not overlap in space, and the action of any finite polynomial in $\boldsymbol{x}$ and $\boldsymbol{p}$ on any of them does not change that fact, these terms will vanish. Therefore, the expectation value of any polynomial in $\boldsymbol{x}$ and $\boldsymbol{p}$ will not depend on α.

** Note that in our case $e^{i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar}\Psi_1=\Psi_2$. Therefore $e^{-i\alpha}\langle\Psi_2\, e^{i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar}\Psi_1\rangle=e^{-i\alpha}\langle\Psi_1\Psi_1\rangle=e^{-i\alpha}/2$

Note also that Ψ_α is a non-analytic function of x and therefore $\langle e^{i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar}\rangle=\langle\sum_n (i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar)^n/n!\rangle\neq\sum_n 1/n!\langle(i\boldsymbol{P}\cdot\boldsymbol{L}/\hbar)^n\rangle$

ponents in any direction

$$P_1+P_2=P_1'+P_2'. \tag{17}$$

Consider now the quantities

$$\pi_1(P_1)=\cos(2\pi P_1/P_0) \tag{18}$$

and

$$\pi_2(P_2)=\cos(2\pi P_2/P_0). \tag{19}$$

As indicated above, measuring π_i is the same as measuring $P_i \bmod P_0$.

Momentum conservation implies that *

$$\pi_1\pi_2-\sqrt{1-\pi_1^2}\sqrt{1-\pi_2^2} =\pi_1'\pi_2'-\sqrt{1-(\pi_1')^2}\sqrt{1-(\pi_2')^2} \tag{20}$$

We therefore get after simple manipulation,

$$(\pi_1')^2+(\pi_2')^2-2C\pi_1'\pi_2'=1-C^2, \tag{22}$$

where

$$C=\cos[2\pi(P_1+P_2)/P_0].$$

We see, then, that modular variables satisfy conservation laws of their own. Instead of the conserved unbounded line $P_1+P_2=$const, we have a conserved ellipse as shown in Fig. 5.

If we know the initial values of the modular momentum of the two interacting systems, we may represent their initial state by a point on the conserved ellipse of Fig. 5. As the interaction between the two systems proceeds, the point representing the system will move along the ellipse and eventually come back to its original position. We see then how the periodicity of the non-local phenomena is reflected in the

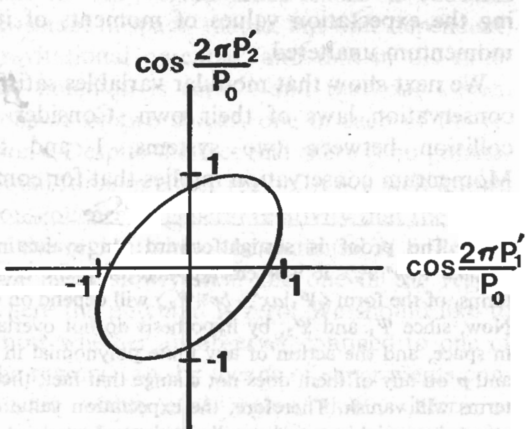

Fig. 5. Conservation law for modular momentum.

* Since $P_1'+P_2'=P_1+P_2$ we have:
$\cos[2\pi(P_1'+P_2')/P_0]=\cos[2\pi(P_1'+P_2')/P_0]$ (21)

conservation laws for the relevant modular variables. We also note that in the classical limit $P_0\to 0$, so that $P \bmod P_0$ changes so rapidly as a function of P as to become entirely unobservable. We see then that it is possible to think about the A-B effect as a non-local exchange of conserved modular variables, something which is obviously completely quantum in origin.

In the rest of this section, and in the next section, we shall consider a number of examples which demonstrate the usefulness of the dynamical approach advocated above.

3.1 *Example: Scattering by electric flux lines*

Consider the effect of a "thin" electric flux line on a charged particle (Fig. 6). Let the electric field producing the flux be given by

$$E_x(x,t)=E_0 \text{ for } |x|\leq \Delta X/2 \text{ and } |t|\leq \Delta t/2$$

and $E_x(x,t)=0$ elsewhere.

The electric flux in this case is given by

$$\Phi=E_x c\Delta x\Delta t \tag{23}$$

Let us calculate the scattering cross section for this case assuming for simplicity that the particle has been prepared initially in an eigenstate of momentum $P_x=P_y=P_z=0$. It is straightforward then to show that, provided Δt is sufficiently small, so that the impulse

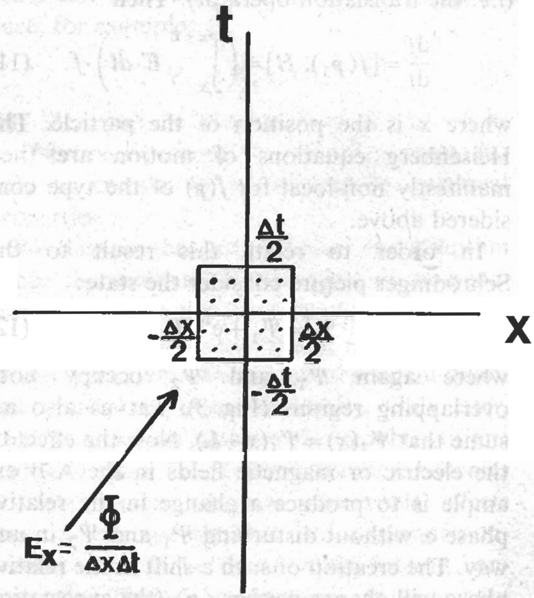

Fig. 6. Electric flux confined in a small space time region.

approximation is valid, the probability amplitude for the momentum of the particle after the flux is turned off is given by*

$$f(k)=\frac{\sin \pi\alpha}{k}+\Delta x\frac{\sin[(k\Delta x-\Phi)/2]}{k\Delta x-\Phi} \qquad (24)$$

where

$$\alpha=\Phi/\Phi_0,\ k=p/\hbar,\ \Phi_0=ch/e. \qquad (25)$$

This result may be understood using the dynamical insight we have gained. We notice that $f(k)$ is made up of two terms. The first term is periodic in the flux and is independent of Δx. This term evidently represents the non-local exchanges of the modulo momentum. The second term, which is small in magnitude (of the order of Δx), accounts for a large exchange of momentum (of the order of $E_0 c\Delta t=\Phi/\Delta x$). This term evidently represents the rare occasions when the particle interacts locally with the field. Consider now the scattering of the particle from a periodic array of idential flux "lines". If we choose

$$\Phi=n\Phi_0+\varepsilon\Phi_0 \qquad (26)$$

where $\varepsilon \ll 1$, we find that the resulting momentum transfer equals $\varepsilon\hbar/L$, where L is the distance between neighboring flux lines.**

Thus for negative ε the particle will be scattered in a direction *opposite* to that of the force. Of course the Ehrenfest theorem tells us that the average momentum shift must be in the same direction as that of the force. Indeed, very rarely, with probability proportional to $\Delta x/L$, a local collision will occur and the large momentum exchange that takes place (this time in the "right" direction) compensates for all the rest of the collisions that went in the opposite direction.

This example demonstrates a general situation that prevails in quantum interactions which vary sharply in space and in time. The effect of such interactions is generally composed of two parts. The first part represents the non-local exchange of modular conserved quantities, and is therefore periodic as a function of the strength of the interaction. This periodicity allows us to choose the interaction strength so that the resulting effect will go in the opposite direction to that predicted by classical theory. The second part corresponds to the local exchanges. By choosing interactions that are sufficiently sharp, *i.e.*, reducing the regions in which the local forces act (and at the same time increasing their strength so that the non-local exchanges, which are proportional to the appropriate flux, remain unaltered) we may reduce the probability for local exchanges at will. Note, though, that in this case in the rare events where the local interaction does manifest itself, the resulting exchanges will be sufficiently large to compensate for all the other events that went in the "wrong" direction.

3.2 *Non-local phenomena of the second type*

So far we have considered non-local effects unique to gauge theories. These effects were characterized by the property of being mediated locally by gauge dependent potentials. We want to show now that more general effects of a non-local nature exist which are mediated directly by local forces, and which have no classical analogue. We begin with an example. Consider a spin $\frac{1}{2}$ particle confined to a given region of space. The spin of the particle is pointing in the z-direction. An external spatially constant magnetic field in the same direction acts on the particle for some finite time. As is well known, this field will cause the components of the spin in the x-y plane to precess around the z-direction. If this were a classical magnetic moment such a precession could certainly be observed. But for our quantum system, since the direction of the spin in the x-y plane is completely uncertain, there is no local change that is observable, and the only effect of the magnetic field is to produce a change in the phase of the state. Consider now a more elaborate case where our particle is initially prepared in the coherent superposition of two non-overlapping regions of space. Assume that we apply the same magnetic field in one of the regions only. The resulting effect will be to produce a change in the relative

* In the impulse case, $\psi'=e^{-iW(x)\Delta t/\hbar}\psi$, where ψ' is the wave function immediately after the flux is turned off, Ψ is the wavefunction just before it was turned on, and $V(x)$ is the potential, *i.e.*, $eE_x=-\partial V/\partial x$.

** This can be seen by taking the Fourier transform of $e^{-iV(x)/\hbar}$, where $V(x)$ is the potential produced by the array of electric flux lines.

phase of the two packets. This, as we have seen, will also cause an exchange of P mod P_0 where $P_0 = h/L$, L being the distance between the two packets.

Classically, the magnetic field will exchange momentum only with particles which are located in regions where the magnetic field varies in space. But quantum mechanically this is not the case, and the particle can experience the effect of the spatial dependence of the magnetic field even when, as in our example, the variation exists in a region from which the particle is excluded. We see in this example the two ingredients essential for non-local phenomena of the second type:

(1) The local effects of the interaction are completely masked by the quantum uncertainties of the system, so that the system has no local "memory" of the interaction after it is over.

(2) The interaction varies in space in the region separating the two regions in which the particle is located, and this variation is responsible for the non-local exchanges of the relevant modular variables.

Let us consider one more example of such phenomena. Suppose (as shown in Fig. 7) a particle is confined within a box, one wall of which is actually a moveable piston. Suppose, also that at $t=0$ the wave function of the particle is negligible everywhere save within a packet whose width Δx is small compared to the length L of the box, and which is located at the end of the box opposite to the piston. Finally, suppose that the expectation value of the velocity of the particle in the x-direction (see Fig. 7) is sufficiently large that the spread of the packet can be neglected over times of the order of the period T of oscillation of the particle within the box.

Obviously, the energy of such a particle will be rather poorly defined; but, because the particle, after a period T, will return nearly to its original state, its *modular* energy, $e^{iET/\hbar}$, will be quite sharp.

Imagine now that at $t=0$, when the particle in the box is far from the piston, the piston is pushed in a short distance $\delta L \ll \Delta x$ and then brought once again to rest. This can be affected by an elastic collision with an external particle. The modular energy of the particle in the box will now be changed, and it follows from the conservation of modular energy that it must necessarily have exchanged that modular energy with the particle outside. This exchange is obviously another example of Type 2 non-locality. Let's understand the nature of this exchange in greater detail. Suppose that the initial state of the particle in the box satisfies

$$e^{-iET/\hbar}\Psi(0) = \Psi(0), \tag{27}$$

i.e., it is an eigenstate of $E \bmod h/T$ with eigenvalue zero. Because of the displacement of the piston, we now have after one period T,

$$\Psi(T, x) = \Psi(0, x-2\delta L) = e^{-i2p\delta L/\hbar}\Psi(0, x). \tag{28}$$

In our case, where $\Delta x \gg \delta L$, we may write

$$\Psi(T, x) = e^{-i2\langle p\rangle \delta L/\hbar}\Psi(0, x) \tag{29}$$

where $\langle p\rangle$ is the initial expectation value of the momentum.

For $\delta L = nh/2\langle p\rangle$, with n any integer, the modular energy will be entirely unaffected. This periodicity reminds us of what we have encountered before in connection with quantized fluxes in the A-B effect.

It should be noted that in the process just described, the modular energy, $E \bmod h/T$, of the external particle is completely uncertain

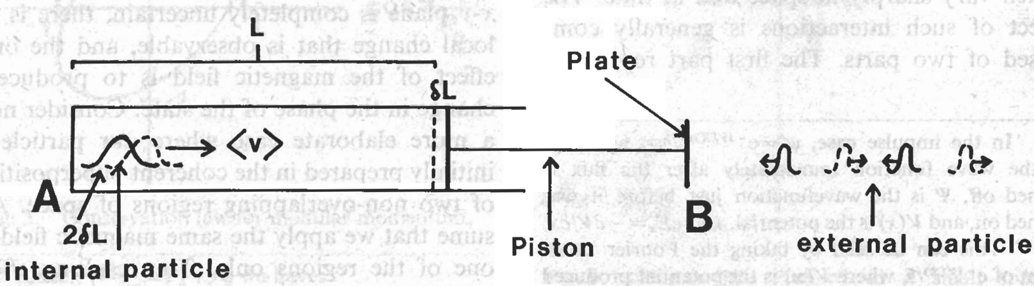

Fig. 7. An example of a non-local exchange of modulo energy.

because the time at which it strikes the piston is specified with an uncertainty much less that T. (This is an essential feature of non-local interactions of this type, the absence of which can be shown to imply acausality.) the effect of this uncertainty is to prevent detection of the exchange of modular energy by observations on the external particle. However, reduction of the uncertainty makes this detection possible. This reduction is achieved by preparing the external particle in a state consisting of two wave packets. If the separation of these packets is given by $L' = \langle v_{\text{ext}} \rangle T$, where $\langle v_{\text{ext}} \rangle$ is the mean velocity of the external particle, there will again be no-local interaction with the internal particle. The result of the collision, then, will be to introduce the relative phase shift whose magnitude is exactly that encountered in eq. (28) and opposite in sign. This exemplifies the exchange of modular variables (in this case energy) subject to an overall conservation law, as discussed above, and is the analogue of local exchanges of conserved quantities.

If the internal particle were initially in a pure energy eigenstate, the interaction with the external particle would no longer be purely non-local, but rather partly local and partly non-local. For the non-local part to be predominant, however, the energy must be large compared to the ground state energy of the box. Just as in the electric flux example of the previous section, there will be a periodic non-local exchange of energy [periodic in $\delta L\sqrt{2mE}/2h$] plus a term representing the usual local exchange. This latter term will be vanishingly small as E becomes large. Again, the periodicity implies that, with probability arbitrarily close to unity, the exchange can go the "wrong way," *i.e.*, can result in the lowering of the internal particle's energy as the piston is pushed in. (The expectation value of its energy will of course rise as a result of those rare, but highly energetic, local exchanges.)

3.3 Semiclassical considerations

Further insight into non-local phenomena may be gained by considering the semi-classical approximation. As we know, in the semi-classical approximation the phase of the wave-function in the Schrödinger representation corresponds to the classical action. Thus we have to find a method of changing the classical action of the set of orbits in a given region of space-time without affecting the orbits themselves. The simplest way of doing this is to introduce a potential which is purely a function of time in the region under consideration since that will augment $S \equiv \int L\,dt$ by an amount $\delta S = \int V(t)\,dt$, (Of course we can achieve a similar result with the aid of a vector potential A.) Thus the A-B effect corresponds to an interaction which produces a change in the action of the classical orbits in a given region of space-time without changing the orbits themselves. (To complete the effect we need to introduce a different change in a second region, which of course implies that we will have non-vanishing fields in the intervening region.)

We call such phenomena non-local phenomena of the first kind. We can now generalize our discussion in the following way. Suppose we introduce in a given region of space a non-zero force field which will modify each orbit in the region for a finite time, but return it afterwards to what it would have been without the field. If we do so for all the orbits in the region, the end result will be to produce no observable change in the final behavior of the classical system. So although we have changed the behavior of the system while the interaction took place, after the interaction is over no "memory" of such changes will persist. What is important for the quantum case, though, is whether any change of *action* occurred due to the interaction, and we shall now show by a simple example that this indeed may be the case.

Consider a situation where we have switched on, and then off again, a given force $F(t)$ which is constant in the whole spatial region. This single impulse will produce a change in both the velocity and the position of each orbit. By introducing a second impulse equal and opposite to the first we may compensate for the change of the velocity but not for the change of positions along each orbit. Let us call this double impulse a "di-pulse". If we now consider adding subsequently an opposite "di-pulse", we will correct also for the change in the position of the orbit. (This consideration is only approximately true if the particle is not free, but we can make this approximation as good as we like by approaching the limit of finite impulses which are sufficiently dense in time.) "Quadra-pulses" of this kind will cause no change in the orbits, but as we shall now see they *will* produce a change in the action of the orbits, and *will* produce a phase change in the corresponding quantum state. For simplicity we shall consider here only a free case, although the result is valid for the more general case if we again make our impulses sufficiently dense in time.

The action of the particle is

$$S=(m/2)\int (v_0+\delta v)^2\, \mathrm{d}t$$

where δv is the change of the velocity due to the effect of the impulses.

$$\delta S=m\int v_0\, \delta\psi\, \mathrm{d}t+(m/2)\int (\delta v)^2\, \mathrm{d}t,$$

wherein the first term vanishes since the impulses are chosen so that $\int \delta v\, \mathrm{d}t=\delta X=0$ where δX is the resulting change in the position of the particle. Thus there is a change in the action which is independent of the orbit, and which will result in a change of phase for the state of the particle if it is confined to the considered region. If we confine our particle to two separated regions of space-time in one of which we introduce these impulses, and in the other of which we do not, we will end up with a result identical to the pure A-B effect. This constitutes an example of non-local phenomenena of the *second* kind (those mediated by *local* interactions which nonetheless produce only non-local changes in the system.) Indeed, if the world were full of such "quadra-pulses", they would produce (at least in the limits of singular impulses) no observable effect classically (just like singular non-quantized lines of flux); but they *would* produce observable effects in the quantum mechanical case.

The final example of non-local phenomenena mediated by local means is of a more general kind and is related in a non-trivial way to the quantum uncertainty principle. Consider again our classical orbits and imagine an interaction that modifies not only the action of the orbit but also the orbit itself. Classically, the effect of the interaction will be observable locally. In the quantum case, though, we can have states, for which the resulting changes of the orbit will be completely masked by the uncertainties of the orbit and the only effect of the added interaction will be to produce a pure phase change of the state. In this case we will end up with a non-local phenomena for a family of states but not for every state. This situation corresponds of course to the two examples of the last section.

§4. Conclusion

In this discussion, I have tried to give a general characterization of non-local phenomena and to outline how one might understand such phenomena in a dynamical way. We have encountered two distinct types of non-local phenomena:

(1) The A-B effect, which is common to all gauge theories, and is characterized by phenomena associated with systems confined to field-free, non-simply connected regions. These were classified as non-local phenomena of type 1.

(2) Effects which arise through local interactions with fields (or other forces), but in such a way as to produce only non-local changes in the systems in question. These were classified as non-local phenomena of type 2.

New types of variables, the modular variables, were introduced, and were shown to be the natural tools for describing non-local dynamics.

References

1) A fairly complete up-to-date list of references concerning the A-B effect is given in S.N.M. Ruijsenaars: Ann. Phys. (N.Y.) **146** (1983) 1.
2) This argument follows closely the arguments of W. H. Furry and N. F. Ramsey: Phys. Rev. **118** (1960) 623.
3) G. Casati: Phys. Rev. Lett. **42** (1979) 1579.
4) D. Wisnivesky and Y. Aharonov: Ann. Phys. (N.Y.) **45** (1967) 479.
5) Y. Aharonov, H. Pendleton and A. Petersen: Int. J. Theor. Phys. **2** (1969) 213.

* * *

C. N. Yang: I have two remarks:

(1) The Bohm-Aharonov effect is the result of a local equation of motion in the Heisenberg representation, but with non-commuting dynamical variables. In this connection, it is *no more* "local" than the barrier penetration phenomena, where the equation of motion is local, but the dynamical variables are non-commuting. If one chooses to call the Bohm-Aharonov effect non-local, then one must also call barrier penetration phenomena non-local.

(2) One lesson we learned in the last some twenty years is that gauge invariance is a most important concept for the physics of fundamental interactions. If one insists on always fixing attention on one gauge, however convenient this might be for each specific problem, one would lose sight of a fundamental characteristics of the interactions. That is, the gauge invariance of all known interactions is what distinguishes them from those not chosen by nature.

A. Shimony: It is usually said that the de Broglie-Bohm hidden variable theories are local, so long as they are applied to a single particle. But doesn't your analysis of the hydrodynamical representation show that non-locality enters even in the single-particle case?

Y. Aharonov: The main lesson of the discussion of the hydrodynamical equations is to show that non-linear equations though local in appearance may hide non-local features. Whether such non-local features may lead to violations of causality in the hidden variable approach to the single particle case is still an open question.

A. Zeilinger: I wish to make two comments with respect to the type 2 non-local experiments you were talking about.

Firstly, I would like to point out, that such an experiment can readily be done in neutron interferometry, if you use properly switched time-dependent magnetic fields. Secondly, I think there exists a conceptual problem with this kind of experiment in that you have to be absolutely sure that the particle spin is in a z-eigenstate in the magnetic field B_z, because otherwise you would be able to observe a local phenomenon, namely Larmor precession.

Y. Aharonov: It is interesting to investigate further the possibility of experimental consequences of type 2 non-locality. As to your second question, one may approach the limit of purely non-local phenomena of type 2, just as one does for the more familiar case of the A-B effect. There one may reduce the probability of local interaction with the flux by decreasing the probability of the particle colliding locally with the flux. Here one has to make the probability of the spin-up state approach 1.

D. M. Greenberger: It seems to me that the conceptual possibility of impenetrable regions of space violates the spirit of both quantum theory and relativity. If the region is impenetrable, you could have line charge inside which you could never know about. This will affect the boundary condition. The only correct boundary condition is the one which agrees with what happens if you drill a hole in the barrier. Otherwise the situation is ambiguous, according to Wu-Yang.

Also, in such regions, the particles on the edges must start to move before waves reach them, in order to cancel out the effects inside. This violates relativity.

Y. Aharonov: I agree.