The analysis of Hardy's experiment revisited

Lev Vaidman

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel-Aviv University, Tel-Aviv 69978, Israel.

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Cohen and Hiley [Phys. Rev. **A 52**, 76 (1995)] have criticized the analysis of Hardy's gedanken experiment according to which the contradiction with quantum theory in Hardy's experiment arises due to the failure of the "product rule" for the elements of reality of pre- and post-selected systems. It is argued that the criticism of Cohen and Hiley is not sound.

Cohen and Hiley [1] analyzed the discussions [2–5] which followed a gedanken experiment proposed by Hardy [6]. In his original letter Hardy claimed that this gedanken experiment proves the impossibility of construction of Lorentz-invariant elements of reality. I pointed out [5] that Hardy's proof relies on the "product rule" of elements of reality. This rule does not hold for pre- and post-selected quantum systems considered in Hardy's experiment. Cohen and Hiley, however, claimed that the contradiction in the analysis of Hardy's experiment is a consequence of the well-known noncovariance of the reduction postulate. They claimed that the contradiction has nothing to do with realism and that the non-applicability of the product rule is essentially unrelated to this (the detailed analysis of the latter issue appears in Ref. [7]).

Based on Cohen [8], Cohen and Hiley also claimed that my analysis of Hardy's example is not valid because it "makes incorrect use of the formula of Aharonov, Bergmann and Lebowitz" [9]. I show that Cohen's argument is not sound elsewhere [10,11].

In this comment I will argue that, contrary to the Cohen and Hiley claims, the arguments of Hardy went beyond showing the noncovariance of the projection postulate and that the contradictions obtained on the basis of Hardy's experiment do not hold if we do not accept the product rule or the closely connected "and rule" [12].

A covariant reduction of quantum states (i.e. collapse) is not a prerequisite for a Lorentz-invariant theory consistent with quantum mechanics. Indeed, there are several proposed interpretations of quantum theory without collapse: the Many-Worlds Interpretation [13], various hidden variable theories such as the Causal Interpretation [14], and some constructions based on the two-state vector formalism [15,16]. Hardy did not consider the many-worlds option and he overlooked the possibilities of the two-state vector approach, but he did consider hidden variable theories. Clearly, the noncovariance of the collapse does not affect the Lorentz-invariance of such (non-collapse) theories. Therefore, even if the noncovariance of the collapse is established, the question of whether a Lorentz-invariant quantum theories exist remains open. Hardy's work was a step towards answering this question.

In order to make a comparison between the arguments

demonstrating the non-covariance of the collapse and Hardy's argument I will analyze a simple example. Following Aharonov and Albert [17], consider a particle located in three separate boxes A, B and C. The particle is prepared in the initial state

$$\frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle), \tag{1}$$

where $|A\rangle$ signifies a particle located in box A, etc. Assume that in one Lorentz frame boxes A and C are opened simultaneously and the particle is found in C. Then, in a Lorentz frame in which the measurement in box A is performed first, the evolution of the state is:

$$\frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle) \ \rightarrow \ \frac{1}{\sqrt{2}}(|B\rangle + |C\rangle) \ \rightarrow \ |C\rangle. \tag{2}$$

However, in a Lorentz frame in which box C is opened first, the evolution is:

$$\frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle) \ \rightarrow \ |C\rangle. \tag{3}$$

Therefore, in one frame the particle was located in two boxes for some period of time while in another it was never located in two boxes (it was located in all three and then just in one). This radical difference in the descriptions of the two Lorentz observers indicates the non-covariance of the collapse.

Let us turn now to Hardy's argument. In order to make the comparison more clear I will consider the analog of Hardy's original proposal for a system of two spatially separated spin-1/2 particles. Assume that the two particles are prepared in the initial state (in the spin z representation)

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}}(\ |\uparrow\rangle_1|\uparrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2 + |\uparrow\rangle_1|\downarrow\rangle_2 \ ). \tag{4}$$

Then, in a given Lorentz frame, simultaneous measurements of both spins in x direction are performed and found to be "down", i.e. the system is found in the state

$$|\Psi_2\rangle = \frac{1}{2}(|\uparrow\rangle_1 - |\downarrow\rangle_1)(|\uparrow\rangle_2 - |\downarrow\rangle_2). \tag{5}$$

(Since $\langle\Psi_2|\Psi_1\rangle \neq 0$ this result is possible.) Straightforward calculations show that in this situation the Lorentz

observer who sees the measurement of particle 1 being performed first concludes that after this measurement and before the measurement performed on particle 2 the state of particle 2 is $|\downarrow\rangle_2$. Similarly, the observer who sees the measurement on particle 2 being performed first concludes that the state of particle 1 after this and before the measurement performed on particle 1 is $|\downarrow\rangle_1$. The time evolutions of the state of the two particles according to the two observers are:

$$|\Psi_1\rangle \rightarrow \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 - |\downarrow\rangle_1)|\downarrow\rangle_2 \rightarrow |\Psi_2\rangle, \qquad (6)$$

$$|\Psi_1\rangle \rightarrow \frac{1}{\sqrt{2}}|\downarrow\rangle_1(|\uparrow\rangle_2 - |\downarrow\rangle_2) \rightarrow |\Psi_2\rangle. \qquad (7)$$

The descriptions of the two observers are different, but not so radically different as in the previous example. Anyway, this difference was not used as a basis of the arguments leading to contradiction in Hardy's experiment.

In order to reach the contradiction, Cohen and Hiley *combined* the statements of one Lorentz observer about particle 1 and the other observer about the state of particle 2 and concluded that the state of the two-particle system before the measurements of the x spin components was

$$|\Psi\rangle = |\downarrow\rangle_1|\downarrow\rangle_2. \qquad (8)$$

However, the initial state is orthogonal to this state, $\langle\Psi|\Psi_1\rangle = 0$, and this is the contradiction of Cohen and Hiley. [18]

In the process of combining the statements of the two Lorentz observers Cohen and Hiley used a variation of the "and rule" [12]: if $A = a$ is an element of reality and $B = b$ is an element of reality then $\{A = a$ and $B = b\}$ is also an element of reality. In this case "$A = a$" is replaced by the statement about the state of particle 1, $|\Psi\rangle_1 = |\downarrow\rangle_1$, and "$B = b$" is replaced by $|\Psi\rangle_2 = |\downarrow\rangle_2$. The "and rule" is closely connected to the product rule and it also *does not hold* for the pre- and post-selected systems [12]. Thus, it is not surprising that adopting the "and rule" leads to a contradiction in the analysis of Hardy's experiment.

Beyond showing that the core of these contradictions lies in the failure of the product rule and the "and rule", it is important to analyze the possibility of relaxing the requirement that these rules are fulfilled (thus allowing the construction of Lorentz-invariant elements of reality for pre- and post-selected quantum systems [5]). I argue that the failure of the "and rule" is indeed what happens in Hardy's experiment. The physical (operational) meaning of the statement that the state of particle 1 is $|\Psi\rangle_1 = |\downarrow\rangle_1$ is: a measurement of the x spin component must yield $\sigma_{1x} = -1$. This statement must be supplemented by the following condition: "provided we do nothing which might disturb the state of the particle". In Hardy's experiment this condition means, in particular, that we do not make a measurement of the x spin component of particle 2. Similarly, a measurement of the x spin component of particle 2 must yield $\sigma_{2x} = -1$, provided we do not measure the spin of particle 1. Obviously, combining this statements does not tell us what will be the results of the x spin component measurements performed on both particles together.

Abandoning the "and rule" allows us to provide a Lorentz-invariant description of a pre- and post-selected quantum system [15,16]. Although we cannot combine the statements $|\Psi\rangle_1 = |\downarrow\rangle_1$ and $|\Psi\rangle_2 = |\downarrow\rangle_2$ in a naive way, we can find a certain operational meaning of these two statements together. It has been shown [19] that in any situation in which the outcome of a standard measurement of a variable is known with certainty, the outcome of the *weak measurement* [20] of this variable must yield the same value. The weak measurement (which is a standard measuring procedure with weakened coupling) in an appropriate limit does not change the quantum state of the system. Therefore, a weak spin measurement on one particle in Hardy's experiment does not affect the state of the two-particle system and, in particular, the state of the other particle. Consequently, we can perform weak measurements of spin x components of both particles and the statements $|\Psi\rangle_1 = |\downarrow\rangle_1$ and $|\Psi\rangle_2 = |\downarrow\rangle_2$ remain true. These statements allow us to deduce the outcomes of these weak measurements, and in this way our Lorentz-invariant description tells us the results of actual experiments. The limiting operational sense of weak measurements is due to the fact that usually (and in particular in Hardy's experiment) an ensemble of pre- and post-selected quantum systems is needed for obtaining a dispersion-free outcome of a weak measurement [21].

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[18] Hardy formulates this contradiction in a slightly differ-
ent way: one Lorentz observer concludes that the value of the projection operator on the state $|\downarrow\rangle_1$ is 1 with certainty, $\mathbf{P}_{|\downarrow\rangle_1} = 1$, the other concludes that $\mathbf{P}_{|\downarrow\rangle_2} = 1$, but $\mathbf{P}_{|\downarrow\rangle_1}\mathbf{P}_{|\downarrow\rangle_2} = 0$. Note that there is no analog of this contradiction in the example of three boxes: $\mathbf{P}_{|B\rangle}+\mathbf{P}_{|C\rangle} = 1$, with $\mathbf{P}_{|C\rangle} = 1$ leads to the correct $(\mathbf{P}_{|B\rangle}+\mathbf{P}_{|C\rangle})\mathbf{P}_{|C\rangle} = 1$.

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