Measurement of an integral of a classical field with a single quantum particle

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A method for measuring an integral of a classical field via local interaction of a single quantum particle in a superposition of 2^N states is presented. The method is as efficient as a quantum method with N qubits passing through the field one at a time and it is exponentially better than any known classical method that uses N bits passing through the field one at a time. A related method for searching a string with a quantum particle is proposed.

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For the past two decades, we are witnessing dramatic growth of research in the field of quantum information: analysis of information tasks that can be performed more efficiently using quantum devices, for example, fast computation [1,2], fast searching [3,4], and efficient solution of some communication complexity tasks [5]. Since quantum "hardware" used for storing, transmitting, and manipulating information is usually very different from its classical counterpart, there is no unique way to make the comparison between quantum and classical systems. It has become customary to measure quantum and classical information systems by comparing the number of basic data storage units—namely, qubits and bits, respectively—needed for a particular task. However, other aspects may also prove to be important. For example, due to difficulties in arranging direct photon-photon interactions, an extensive research of what can be achieved using linear optical devices was done [6]. Thus, the number of qubits stored in the Hilbert space of the quantum system performing the information task is not always the only (or the best) measure by which we can evaluate the efficiency of a quantum system. Depending on the possibility of practical applications, various quantum schemes might have particular advantages.

Grover's fast search algorithm [3,4] which uses N qubits can be performed with a single particle with 2^N states [7]. Meyer [8] suggested that it can be done for other tasks, too, and in this paper we present such modification for a recently proposed task of measuring an integral of a classical field using quantum devices.

Recently, a quantum method using a single qubit for measuring the parity of an integral of a classical field,

$$I = \int_A^B \phi(x)dx, \tag{1}$$

provided it takes on only positive integer values, has been suggested [9]. This method was generalized, by Vaidman and Mitrani (VM) [10], to compute the value of the integral itself, using N qubits represented by N spin-$\frac{1}{2}$ particles (or any other two-level quantum systems) which are sent one at a time through the field. Furthermore, the VM method is applicable when the integral may take on noninteger values. The precision of this method turns out to be exponentially better than any known classical method which uses N bits sent one at a time.

We will describe how a single (spin-zero) particle, which passes only once through the field, can be used to evaluate the integral of the field with the same precision as the VM method. We let the particle be in $K=2^N$ distinct sites, so our method has exponentially increasing requirements for space, time, and precision [11,12]. However, it still has a potential for practical advantage; see the discussion of other quantum methods [13,14].

We outline the algorithm in what follows. The particle is initially prepared to be in a superposition of equal amplitudes and vanishing relative phases, over the $K=2^N$ consecutive separate sites,

$$|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{K}}\sum_{k=1}^{K}|k\rangle. \tag{2}$$

Next, we send this "train of amplitudes" through the field with constant velocity (see Fig. 1).

We arrange a local field-particle interaction of the form

$$H_{\text{int}} = g(x,t)\phi(x), \tag{3}$$

in such a way that the strength of the coupling of the field to the kth part of the particle is proportional to the index number k:

$$g(x_k(t),t) = \frac{k}{K\alpha}, \tag{4}$$

where $x_k(t)$ is the location of the kth part at time t and α is a parameter that we fix depending on the given information about possible values of the integral of the field. After the particle completes its passage through the field, its final state (due to the interaction) is

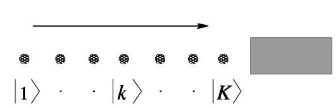

FIG. 1. The "particle train" passes through the field.

$$|\Psi_{\text{fin}}(I)\rangle = \frac{1}{\sqrt{K}}\sum_{k=1}^{K} e^{-i2\pi kI/K\alpha}|k\rangle. \tag{5}$$

For the special case of $I=\alpha m$, where $m=0,1,\ldots,K-1$, we obtain K mutually orthogonal states. These K states represent a basis of the Hilbert space of the particle. Thus, a measurement in this basis yields the correct value of m with probability 1, exactly like the VM method does with $N=\log_2 K$ particles.

The VM method [10] also provides an answer with a good precision for a more general case when I is not necessarily a multiple of α. In this case, the measurement always yields one of the discrete values $\widetilde{I}=\alpha m$, and the probability for the error $\delta I=\widetilde{I}-I$ is

$$p(\delta I) = \prod_{n=1}^{N} \cos^2 \frac{\delta I\pi}{2^n\alpha}. \tag{6}$$

In our algorithm we also get one of the values $\widetilde{I}=\alpha m$, and the probability for the error is given by the squared norm of the scalar product of the states corresponding to I and $\widetilde{I}$:

$$p(\delta I) = |\langle\Psi(\widetilde{I})|\Psi(I)\rangle|^2 = \frac{\sin^2(\delta I\pi/\alpha)}{4^N\sin^2(\delta I\pi/2^N\alpha)}. \tag{7}$$

Although expressions (6) and (7) look different, they are, in fact, identical. This can be checked in a straightforward manner by mathematical induction on N. The equality is not a coincidence. In fact, from the mathematical point of view, the two methods are isomorphic. We can make the correspondence between the state $|k\rangle$ and a state of N spin-$\frac{1}{2}$ particles which "writes" the number k in a binary form with $|{\uparrow}\rangle\equiv 0$ and $|{\downarrow}\rangle\equiv 1$. We arrange the interaction between the spins and the field such that the spin corresponding to the jth digit accumulates the phase $-2\pi 2^j I/K\alpha$ when the spin is "down" and zero phase when the spin is "up." In this way the overall phase of N particles in a state corresponding to state $|k\rangle$ will be exactly as in our case: $-2\pi kI/K\alpha$. Thus, if we start with N spins originally pointing in the x direction, i.e., in the state $(1/\sqrt{2})(|{\uparrow}\rangle+|{\downarrow}\rangle)$, then we obtain the state (5) after the interaction, with the only change that $|k\rangle$ represents a corresponding state of N spin-$\frac{1}{2}$ particles. The interaction that leads to the phase $-2\pi 2^j I/K\alpha$ when the spin j is "down" and no phase if the spin is "up" is exactly the magnetic field in the z direction of the VM method. Therefore, mathematically, the two methods are equivalent. The implementation is of course different. It depends on the physical system which is easier: sending N spins one at a time or sending the train of 2^N wave packets.

The function $p(\delta I)$ is exactly the interference pattern of $K=2^N$ slits (see Fig. 2). It becomes well localized with large K, but it is periodic with period αK. In fact, what is measured is $I\text{mod}(\alpha K)$ and the error should be understood as $(\widetilde{I}-I)\text{mod}(\alpha K)$. Following VM, we consider the situation in which I is of the order of $M=\alpha K/10$, so we can neglect the complications following from the periodicity of the function $p(\delta I)$.

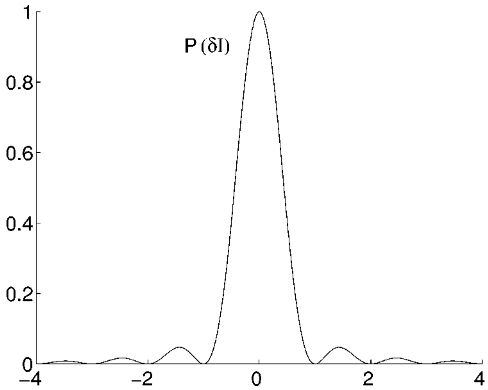

FIG. 2. The probability of the error $p(\delta I)$ for the quantum methods for $N=7$. (In the figure, δI is in units of α.) The outcome of the measurement $\widetilde{I}$ may be one of the values αm, so in a particular experiment δI may obtain only a discrete value $\alpha n+I\text{mod}\alpha$.

The uncertainty of the measurement can be characterized as the standard deviation

$$\Delta I = \sqrt{\langle I^2\rangle - \langle I\rangle^2} \simeq \frac{10M}{\sqrt{2}\pi}\frac{1}{\sqrt{2^N}}. \tag{8}$$

It is also useful to compute another measure of uncertainty, namely, the mean absolute deviation of the measured value,

$$\Delta' I = \langle|\delta I|\rangle \simeq \frac{10M}{2\pi^2}\frac{\ln 2^N}{2^N}. \tag{9}$$

The uncertainty of the corresponding classical method, described in [10], in which N bits are sent one at a time through the field, is of the order of $1/\sqrt{N}$, i.e., it is exponentially larger than the uncertainty in quantum methods. If we remove the constraint of sending bits one after the other, we can construct a much better classical method, but still there is some advantage for the quantum methods. In this case the N bits are sent together and they function as a counter which can go up to 2^N. If we arrange that the counter "clicks" while moving through the field with probability

$$dp = \alpha\phi(x)dx, \tag{10}$$

then the resulting standard deviation $\Delta I_{\text{cl}} \simeq \sqrt{10MI/2^N}$ is of the same order as the standard deviation in quantum methods (8). However, the average of the absolute value of the error

$$\Delta' I_{\text{cl}} = \langle|\delta I_{\text{cl}}|\rangle \simeq \sqrt{\frac{2}{\pi}}\Delta I_{\text{cl}} = \sqrt{\frac{20MI}{\pi 2^N}} \tag{11}$$

turns out to be larger than that of the quantum methods (9).

It is interesting to note that a classical algorithm can achieve the same precision by sending the bits one by one, when local memory is allowed. We first start with a particle (a "marker") which goes through the field and occasionally leaves marks with the same probability law (10) as our N-bit counter. Then, we use our bits to count the marks. The counting of the marks can be done in the following way. At the beginning, all bits are initialized to 0. The bits go one after the other along the path. They all behave according to the following rule: when a bit in a state 0 meets a mark, it erases the mark and flips to 1, while when a bit in a state 1 meets a mark, it leaves the mark undisturbed and flips to 0. It is easy to see that the final state of the bits after they all pass through the marked path is the binary representation of the total number of marks created by the marker.

In our method all parts of the wave function of the particle pass through all points of the field. Can we get some information when different parts of the wave of the particle pass through only parts of the field? We cannot find the integral of the field in this way. But there is a specially tailored task of a similar type which can be accomplished with a single quantum particle. The classical solution of this task requires a large number of bits.

Consider $K=2^N$ local classical bits which we want to read. There are 2^K possible strings $\{a_k\}$, but in our special task we consider a situation in which it is known that our set of bits can be in one of $N+1$ specific strings. Whatever the strings are, in order to find the string, the number of bits we have to approach is larger than $\log_2 N$ because these bits have to specify the chosen string. Since in this scenario each particle (and in the quantum analog each part of the particle wave) approaches only one bit, classically, we need at least $\log_2 N$ particles. We will show that for a specific set of strings we need just one quantum particle to achieve this goal.

For $K=16$ our set of strings is

$$\begin{array}{cccccccccccccccc} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \end{array}. \tag{12}$$

The general rule is clear from the example. In the nth string, the set of $K/2^{n-1}$ bits 0 is followed by the same number of bits 1, which is followed again by the same number of bits 0, etc., until the string ends.

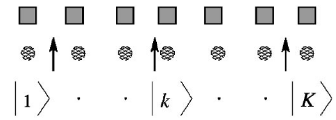

FIG. 3. A single quantum particle reads a string of K bits. Each part of the superposition of the particle passes through the location of one of the bits.

In our quantum method, we again use a single quantum particle prepared in a superposition of K states without relative phase (2). Each part of the superposition passes through a location of one of the bits, Fig 3. The interaction (as in the Bernstein-Vazirani problem [15]) is such that it acquires phase π if the bit is 1 and 0 if the bit is 0. It is easy to see that for different strings from our special set we obtain in this way mutually orthogonal states. Thus, we have shown that a single quantum particle can read reliably a 2^N-bit string provided it is one out of the particular $N+1$ strings. Using classical devices, for this task we need more than $\log_2 N$ bits.

It seems that technology today is not at the stage of building a quantum device for the proposed task which works better than its classical counterpart. However, experiments, similar to those that show a proof of principle for operating a quantum computer are certainly capable of showing the proof of principle of the results presented here. Since the methods presented here are applicable for measurement of an integral of a quantum field, or the sum of values of registers in a quantum computer, they might be useful for designing future quantum information devices.

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