

Strict bounds on the Franson inequality

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An inequality, recently proposed by Franson [Phys. Rev. A **54**, 3808 (1996)], is analyzed and improved. The inequality connects the change of the expectation value of an observable with the uncertainty of this observable. A strict bound on the ratio between these two quantities is obtained. [S1050-2947(98)04503-X]

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In a recent paper Franson [1] developed a quantitative expression that characterizes the fact that a quantum observable cannot change its expectation value without having some quantum uncertainty. The simplest explanation of this property is that in order for a quantum system to go from one eigenvalue of an observable to another it must evolve through a superposition of the eigenstates that is characterized by nonzero uncertainty of the observable. Franson proposed to consider the change of an observable Q after it has been measured such that the initial state of the system at time $t=0$ is an eigenstate of the observable. Then, at any moment of time t in the period τ_1 ,

$$\tau_1 = \frac{\hbar}{\sqrt{2}\Delta E}, \quad (1)$$

where ΔE is the energy uncertainty of the system, the following inequality is fulfilled:

$$|\delta\langle Q \rangle| \leq \Delta Q. \quad (2)$$

Here ΔQ is the uncertainty at time t ,

$$\Delta Q = \sqrt{\langle \Psi(t) | Q^2 | \Psi(t) \rangle - \langle \Psi(t) | Q | \Psi(t) \rangle^2}, \quad (3)$$

and $\delta\langle Q \rangle$ is the change in the expectation value,

$$\delta\langle Q \rangle = \langle \Psi(t) | Q | \Psi(t) \rangle - \langle \Psi(0) | Q | \Psi(0) \rangle. \quad (4)$$

If an inequality expresses a basic physical law, there must be cases where it is saturated; if not, it must be replaced by a stricter inequality that does saturate. The Franson inequality as it is presented in his paper cannot reach the equality (except at $t=0$). The purpose of this paper is to expand Franson's bound such that the revised inequality cannot be replaced by any stricter inequality.

There are two elements by which we expand the Franson inequality. First, we replace his approximate calculations by exact calculations and obtain, instead of Eq. (1), a larger period of time τ_2 for which the inequality (2) is fulfilled. Then, for time $t = \tau_2$ the inequality can be saturated. However, for any intermediate time, $t \in (0, \tau_2)$, the inequality cannot be saturated. In order to correct this we introduce time explicitly into the inequality.

The Franson inequality is closely connected with a certain type of time-energy inequality that constrains the time of the evolution of a system to an orthogonal state:

$$t \geq \frac{\hbar}{4\Delta E}. \quad (5)$$

Although it has a similar form, it is conceptually different from the Heisenberg uncertainty relations for position and momentum.¹ There are many derivations of this result. As far as we know the first derivation was given by Mandelstam and Tamm [3], and a very simple derivation can be found in Vaidman [4]. We will apply this result for deriving a strict bound on the Franson inequality. A more general form of the time-energy inequality that we will need here is

$$|\langle \Psi(t) | \Psi(0) \rangle| \geq \cos\left(\frac{\Delta E t}{\hbar}\right), \quad (6)$$

which is valid for $t \in [0, \pi\hbar/2\Delta E]$, where ΔE is the energy uncertainty of the system. This inequality is quoted by Yu [2] [his Eq. (1)] and it can be obtained, e.g., from Eq. (8) of Ref. [4].

Let us turn to finding the optimal bound on the period of time for which the Franson inequality (2) is fulfilled. This problem is equivalent to finding the minimal time for which $|\delta\langle Q \rangle| = \Delta Q \neq 0$. Without loss of generality we can assume that the eigenvalue of Q at the initial time is zero, i.e., $|\Psi(0)\rangle = |Q=0\rangle \equiv |0\rangle$. At time t the state of the system can be expressed as

$$|\Psi(t)\rangle = \alpha|0\rangle + \beta|1\rangle, \quad (7)$$

where $|1\rangle$ is orthogonal to $|0\rangle$. For convenience, we define $|1\rangle$ in such a way that β is real and positive. Then, applying definitions (3) and (4) we obtain

$$\Delta Q = \sqrt{\beta^2\langle 1 | Q^2 | 1 \rangle - \beta^4\langle 1 | Q | 1 \rangle^2}, \quad (8)$$

$$\delta\langle Q \rangle = \beta^2\langle 1 | Q | 1 \rangle. \quad (9)$$

From $|\delta\langle Q \rangle| = \Delta Q$ follows:

$$\beta^2 = \frac{1}{2} \frac{\langle 1 | Q^2 | 1 \rangle}{\langle 1 | Q | 1 \rangle^2}. \quad (10)$$

¹Recently, Yu [2] described an analog of this time-energy inequality for position and momentum.

Since $\langle 1|Q|1 \rangle^2$ is always smaller than or equal to $\langle 1|Q^2|1 \rangle$, the minimal value for β is $1/\sqrt{2}$. For this value of β we obtain

$$|\langle \Psi(t) | \Psi(0) \rangle| = |\alpha| = \sqrt{1 - \beta^2} = \cos \frac{\pi}{4}. \quad (11)$$

From the time-energy relation (6) it follows that the minimal time [which corrects Franson's bound (1)] is

$$\tau_2 = \frac{\pi \hbar}{4 \Delta E}. \quad (12)$$

Since $1/\sqrt{2} < \pi/4$, the time limit we found is indeed larger than that of Franson [Eq. (1)]. Note that this is half the minimal time of an evolution to an orthogonal state.

Consider an example of a spin- $\frac{1}{2}$ particle precessing in a magnetic field. Let us assume that the initial state is $|\uparrow\rangle$, the ‘‘up’’ state in the z direction, and that the magnetic field is in the y direction. Then, the time evolution can be expressed in the following form:

$$|\Psi(t)\rangle = \cos\left(\frac{\Delta E t}{\hbar}\right) |\uparrow\rangle + \sin\left(\frac{\Delta E t}{\hbar}\right) |\downarrow\rangle. \quad (13)$$

Therefore,

$$\delta\langle\sigma_z\rangle = \cos\left(\frac{2\Delta E t}{\hbar}\right) - 1, \quad (14)$$

$$\Delta\sigma_z = \sin\left(\frac{2\Delta E t}{\hbar}\right). \quad (15)$$

Thus we see that $|\delta\langle\sigma_z\rangle| \leq \Delta\sigma_z$ for $t \leq \tau_2$. The equality is reached for $t = \tau_2$.

The same bound is obtained for any other two-level system or even for any system with higher-dimensional Hilbert space in case the quantum state of the system moves only inside a two-dimensional subspace during its evolution. Introducing any additional state will invariably increase the minimal time for reaching the equality $|\delta\langle Q \rangle| = \Delta Q$.

It is interesting to consider the connection between $\delta\langle Q \rangle$ and ΔQ when we do not impose the initial condition $\Delta Q = 0$. We ask what is the minimal period of time during which the change in the expectation value reaches the value of the maximal uncertainty during this period of time.

Consider our example of a spin- $\frac{1}{2}$ particle. A simple analysis shows that the minimal time to reach the equality $|\delta\langle Q \rangle| = \Delta Q$ is obtained for the evolution between the states

$$|\Psi_{\text{in}}\rangle = \cos\left(\frac{\pi}{6}\right) |\uparrow\rangle + \sin\left(\frac{\pi}{6}\right) |\downarrow\rangle, \quad (16)$$

$$|\Psi_f\rangle = \cos\left(\frac{\pi}{3}\right) |\uparrow\rangle + \sin\left(\frac{\pi}{3}\right) |\downarrow\rangle. \quad (17)$$

This time is

$$\tau_3 = \frac{\pi \hbar}{6 \Delta E}. \quad (18)$$

In this case $|\delta\langle\sigma_z\rangle| = \max(\Delta\sigma_z) = 1$ and the maximum uncertainty is obtained at the middle point of the evolution.

However, an example in which only two states are involved is not the optimal one. A system of N spin- $\frac{1}{2}$ particles precessing in a magnetic field yields a smaller time for reaching the equality $|\delta\langle\Sigma_i(\sigma_z)_i\rangle| = \Delta[\Sigma_i(\sigma_z)_i]$. The minimal time for this is

$$\tau_4 = \sqrt{N} \arcsin\left(\frac{1}{2\sqrt{N}}\right) \frac{\hbar}{\Delta E}. \quad (19)$$

At the limit of $N \rightarrow \infty$ we reach the minimal time

$$\tau_5 = \frac{\hbar}{2\Delta E}. \quad (20)$$

The same minimal time is obtained in the most natural example of comparison between $\delta\langle Q \rangle$ and ΔQ . Consider a free particle of mass m in one dimension in a minimal uncertainty Gaussian wave packet that moves with high velocity v , such that the expectation value of the particle's momentum is much larger than its uncertainty, $\langle p \rangle \gg \Delta p$. Then, since $E = p^2/2m$, we obtain

$$\Delta E \sim \frac{\langle p \rangle \Delta p}{m}. \quad (21)$$

Taking into account the Heisenberg relation for minimal uncertainty, $\Delta x \Delta p = \hbar/2$, we find the time for which $\delta\langle x \rangle = \Delta x$:

$$\frac{\Delta x}{v} = \frac{\Delta x m}{\langle p \rangle} = \frac{\Delta x \Delta p}{\Delta E} = \frac{\hbar}{2\Delta E}. \quad (22)$$

(We have disregarded the change of the uncertainty in position during this time because it is negligible in our case for which $\langle p \rangle \gg \Delta p$.)

The period τ_5 is a strict bound for the minimal period of time during which the change in the expectation value reaches the value of the maximal uncertainty. This result can be obtained immediately from the Heisenberg relation:

$$\Delta E \Delta Q \geq \frac{1}{2} |\langle [H, Q] \rangle| = \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|. \quad (23)$$

If ΔQ is essentially constant during the process, we find that the equality $|\delta\langle Q \rangle| = \Delta Q$ is reached during the time τ_5 .

The strict bound τ_5 and the bounds τ_3 and τ_4 are bounds for a different problem from the one Franson proposed. The novelty of his inequality is in considering an evolution starting from an eigenstate of an observable. For such a problem only the Franson bound τ_1 and the improved bound τ_2 are relevant. The bound τ_2 is absolute in the sense that there is no larger bound for which the inequality (2) holds for the whole period $t \in [0, \tau]$. However, even with this exact bound the inequality cannot be considered as a fundamental one because it cannot be saturated for any time except $t=0$ and $t=\tau_2$. In order to find a basic inequality let us return to Eqs.

(8) and (9), but now we will not limit ourselves to the equality $|\delta\langle Q \rangle| = \Delta Q$. By dividing the equations we obtain

$$\frac{|\delta\langle Q \rangle|}{\Delta Q} = \frac{1}{\sqrt{\langle 1|Q^2|1 \rangle / \beta^2 \langle 1|Q|1 \rangle^2 - 1}}. \quad (24)$$

From the time-energy relation (6) we obtain

$$\beta = \sqrt{1 - |\langle \Psi(0) | \Psi(t) \rangle|^2} \leq \sqrt{1 - \cos^2\left(\frac{\Delta E t}{\hbar}\right)} = \sin\left(\frac{\Delta E t}{\hbar}\right). \quad (25)$$

Now, taking into account again that $\langle 1|Q|1 \rangle^2 \leq \langle 1|Q^2|1 \rangle$ we obtain a new inequality:

$$\frac{|\delta\langle Q \rangle|}{\Delta Q} \leq \tan\left(\frac{\Delta E}{\hbar} t\right). \quad (26)$$

This inequality is valid for $t \in (0, \pi\hbar/2\Delta E]$. The inequality (26) is a basic law since it cannot be replaced by a stricter inequality. Indeed, the example of a spin- $\frac{1}{2}$ particle precessing in a magnetic field saturates the inequality: it becomes an equality for the whole period $(0, \pi\hbar/2\Delta E]$.

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