Causality constraints on nonlocal quantum measurements

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Consequences of relativistic causality for measurements of nonlocal characteristics of composite quantum systems are investigated. It is proved that verification measurements of entangled states necessarily erase local information. A complete analysis of measurability of nondegenerate spin operators of a system of two spin-$\frac{1}{2}$ particles is presented. It is shown that measurability of certain projection operators which play an important role in axiomatic quantum theory contradicts the causality principle.

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I. INTRODUCTION

As early as 1931, Landau and Peierls [1] showed that relativistic causality imposes new restrictions on the process of quantum measurement. Although some of their arguments were not precise, it was commonly accepted that we cannot instantaneously measure nonlocal properties without breaking relativistic causality. It was only in 1980 that Aharonov and Albert [2] showed that there are nonlocal variables which can be instantaneously measured without contradicting relativistic causality. The work of Aharonov, Albert, and Vaidman [3] was mainly devoted to finding explicit methods for performing instantaneous nonlocal measurements. Here we derive general properties of all such measurements. Our main result is that instantaneous nonlocal measurement invariably disturbs the measured system in such a way that all local information (except for that which is related to degrees of freedom not involved in the measurement) is erased. That is, when nonlocal measurements are performed on an ensemble of systems, there are limits on the amount of information about the initial state contained in the final state of the ensemble.

Section II is devoted to defining the framework of our research. In Sec. III, we discuss the causality principle and we derive useful equalities which follow solely from causality. In Sec. IV, we define state verification measurement, and, based on the requirement of reliability of the measurement, we prove two theorems about the necessity of erasing local information in verification measurements of entangled states. Section V is devoted to the application of the derived results to analysis of the measurability of spin operators of a system of two spin-$\frac{1}{2}$ particles. It includes a complete analysis of the measurability of nondegenerate operators of this system. In Sec. VI, we investigate the consequences of our results on axiomatic quantum theory, showing that certain *ideal measurements of the first kind* [4] cannot be performed without contradicting the causality principle. A summary of our results concludes the paper in Sec. VII.

II. GENERAL FRAMEWORK

Our present study is restricted to the framework which is generally used for investigating causality constraints on quantum measurements. It was first applied by Bohm and Aharonov [5] and later by Bell [6] for analyzing the Einstein-Podolsky-Rosen argument; and this was the framework in which the first measurable nonlocal variables were found [2]. That is, we consider quantum systems which consist of two distinguishable parts, each localized in a different region of space. We take each region small enough to allow neglecting causality restrictions inside it, but much bigger than a Compton wavelength, so we can neglect relativistic effects such as pair creation. The causality principle enters at the scale of distances between the widely separated parts of the system, while locally we can use the formalism of nonrelativistic quantum mechanics. For example, we shall consider a system of two spin-$\frac{1}{2}$ particles located in remote space regions. Spin components of each particle can be measured by using a Stern-Gerlach apparatus, and these measurements are described by nonrelativistic quantum mechanics.

In this work we study *instantaneous measurements of nonlocal properties.* Following von Neumann, any measurement can be considered as having three stages. The first stage is a preparation of the measuring device. The second stage is an interaction between the measured system and the measuring device. As the result of this interaction, the final state of the measuring device will contain information about the initial state of the system. In the third stage this information is read by observers.

By an *instantaneous* measurement we do not mean that some observer can instantaneously find out the result. Only the second stage of the measurement, the interaction between the measuring device and the system, must be instantaneous (i.e., very short). The measurement as a whole may take a much longer time; the measuring device may have had to be prepared a long time before the interaction and it may take a long time to recover the result.

The measurements considered here are designed to determine nonlocal properties of systems. Nevertheless, the interaction between the measuring device and the measured system need not be nonlocal, since it is possible [2] to measure *nonlocal* properties via *local* interactions. However, in the present work we do not make any particular assumptions about the interaction between the system and the measuring device, apart from unitary of the time evolution.

III. CAUSALITY PRINCIPLE

The causality principle states that observers situated near widely separated parts of a system cannot communicate with one another with superluminal velocity: local interactions performed in one part of the system could not affect the probabilities of the outcomes of local measurements performed on the other part of the system outside the light cone. Consider now the following situation. Our system, consisting of two separate parts, 1 and 2, is prepared, initially, in a state $|\psi\rangle$. At time $t_0-\epsilon$ (with ϵc small compared to the distance between the two parts of the system) some local interaction is performed, say, on part 2 of the system. This interaction is described by a unitary transformation $U^{(2)}$. At time t_0 a nonlocal measurement is performed. More exactly, at t_0 the measuring device interacts with the system and this leads to a unitary transformation U of the state of the composite, i.e., the system and the measuring device. At time $t_0+\epsilon$, a measurement of a local observable $A^{(1)}$ is carried out on part 1. From the causality principle it follows that the probability for any particular outcome of any local measurement, say $A^{(1)}=a$, is independent of the local action on part 2, $U^{(2)}$.

Let us denote the probability for the result $A^{(1)}=a$ of the local measurement in part 1 at time $t_0+\epsilon$, provided the state of the system immediately before the nonlocal measurement performed at t_0 was $|\psi\rangle$, by $p(\psi)$:

$$p(\psi)\equiv \text{prob}(A^{(1)}=a_{\text{at } t_0+\epsilon} \,|\, |\psi\rangle_{\text{at } t_0-\epsilon} ; \; \text{nonlocal measurement}_{\text{at } t_0}) \; . \quad (1)$$

In this compact notation the causality principle yields

$$p(U^{(2)}\psi)=p(\psi) \; . \quad (2)$$

The probability for a given outcome of a measurement is equal to the expectation value of the projection operator onto the corresponding subspace. Let $\mathcal{P}_a^{(1)}$ denote the projection operator onto the subspace corresponding to $A^{(1)}=a$. Then,

$$p(\psi)=\langle\phi|\langle\psi|U^\dagger\mathcal{P}_a^{(1)}U|\psi\rangle|\phi\rangle \; , \quad (3)$$

where $|\phi\rangle$ is the initial state of the measuring device, and where the unitary transformation U describing the nonlocal measurement acts on both states, that of the system and that of the measuring device. In this more explicit notation the causality principle (2) becomes

$$\langle\phi|\langle\psi|U^{(2)\dagger}U^\dagger\mathcal{P}_a^{(1)}UU^{(2)}|\psi\rangle|\phi\rangle = \langle\phi|\langle\psi|U^\dagger\mathcal{P}_a^{(1)}U|\psi\rangle|\phi\rangle \; . \quad (4)$$

By applying Eq. (4) to the states $|\psi_1\rangle+|\psi_2\rangle$ and $|\psi_1\rangle+i|\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are arbitrary, we easily obtain the following generalization of (4):

$$\langle\phi|\langle\psi_2|U^{(2)\dagger}U^\dagger\mathcal{P}_a^{(1)}UU^{(2)}|\psi_1\rangle|\phi\rangle = \langle\phi|\langle\psi_2|U^\dagger\mathcal{P}_a^{(1)}U|\psi_1\rangle|\phi\rangle \; . \quad (5)$$

Note that in the absence of a nonlocal measurement (i.e., in the absence of the operator U), Eqs. (4) and (5) would be trivial due to commutativity of the local operators $\mathcal{P}_a^{(1)}$ and $U^{(2)}$. In general, U does not commute with $U^{(2)}$ or $\mathcal{P}_a^{(1)}$. Thus, Eqs. (4) and (5) represent causality constraints on the possible nonlocal measurements. We shall use them below for deriving necessary properties of nonlocal measurements.

IV. STATE VERIFICATION MEASUREMENTS

More than half a century after the creation of quantum theory there is no clear consensus about the interpretation of its basic concept: a quantum state. Does it represent some kind of reality or is it just a mathematical tool for calculating probabilities? The possibility of instantaneous verification of a quantum state will manifest its physical meaning.

We start by investigating the properties of *state verification* measurements. By a verification of a given state $|\psi_0\rangle$ we understand a measurement which always yields the answer "yes" if the measured system is in the state $|\psi_0\rangle$ and the answer "no" if the system is in an orthogonal state $|\psi_\perp\rangle$. If the initial state is a superposition of $|\psi_0\rangle$ and $|\psi_\perp\rangle$ then the appropriate probabilities for the answers "yes" and "no" will follow from the linearity of quantum theory.

We note that the state verification measurement defined above does not imply anything about the final state of the system, unlike the standard quantum measurements in which the final state is an appropriate eigenstate of the measured operator. In this sense, state verification measurement is more basic.

Causality limitations on quantum measurements were used as an argument against associating physical reality to a quantum state [1]. Indeed, we will show that causality forbids performing state verifications using standard quantum measurements (for example, by measuring the projection operator on $|\psi_0\rangle$). Nevertheless, a method which permits verification of any quantum (even nonlocal) state was found [3]. The method is called "exchange measurement" [7]. The idea of exchange measurement is to make simultaneous short local interactions with parts of the measuring device so that the states of the system and the measuring device will be exchanged. The novel point in this method is that *local* interactions exchange *nonlocal* states. The result of the measurement cannot be read by two local observers; the two parts of the measuring device have to be brought to one place.

Exchange measurements have another very unconventional property: after the measurement, the system ends up in a state $|\psi_{\text{final}}\rangle$ which is completely independent of the initial state of the system, but depends only on how the measurement is designed. Thus, the exchange measurement has the property of erasing from the system all information about the initial state of the system. We emphasize that this erasing of information takes place at the level of an entire ensemble of systems. That is, when exchange measurements are performed on an ensemble of systems, all the systems in the ensemble will end up in the same final state $|\Psi_{\rm final}\rangle$, so that after the measurement there is no trace of the initial sate in the ensemble. This is to be contrasted to what happens in the case of standard quantum measurements. In the latter, after a measurement, each individual system in the ensemble "forgets" its initial state and ends up in an eigenstate of the measured operator. The final ensemble becomes a mixture of eigenstates of the measured operator, but the probabilities with which these eigenstates mix still reflect the initial state—they are just the squares of the absolute values of the projections of the initial state onto the corresponding eigenstates of the measured operator.

We will show that the erasing of information from the system is a generic property of any causal state verification measurement. However, not all information is necessarily erased; the causality requires that *local* information is erased. The probabilities of the outcomes of any local measurement performed after the state verification are independent of the initial state of the system (except possibly for measurements related to degrees of freedom not involved in the state verification).

Let us now enunciate the above property in a more precise form. Consider a measurement designed to verify whether or not a system is in a given state $|\psi_0\rangle$. By choosing appropriate local orthonormal bases in parts 1 and 2, we can decompose $|\psi_0\rangle$ as (Schmidt decomposition)

$$|\psi_0\rangle=\sum_i \alpha_i|i\rangle_1|i\rangle_2 \ . \tag{6}$$

Let us denote by $H^{(1)}$ and $H^{(2)}$ the Hilbert spaces of parts 1 and 2, respectively, and by $H_0^{(1)}$ and $H_0^{(2)}$ the subspaces of $H^{(1)}$ and $H^{(2)}$, which are spanned by the basis vectors $|i\rangle_1$ and $|i\rangle_2$ corresponding to coefficients $\alpha_i\neq 0$. We shall prove that for all initial states $|\psi\rangle$ belonging to the Hilbert space $H_0^{(1)}\otimes H^{(2)}$, the probability $p(\psi)$ for a result of a local measurement performed on part 1, after the state verification of $|\psi_0\rangle$, has no dependence at all on the initial state. In particular, the initial state might be $|\psi_0\rangle$. Using notation (1) we formulate the following theorem.

Theorem 1. If $|\psi\rangle\in H_0^{(1)}\otimes H^{(2)}$, then $p(\psi)=p(\psi_0)$. In general, however, the initial state is not restricted to $H_0^{(1)}\otimes H^{(2)}$; it may be any state belonging to $H^{(1)}\otimes H^{(2)}$. Let us decompose $|\psi\rangle$ as

$$|\psi\rangle=\alpha|\psi'\rangle+\beta|\psi''\rangle \ , \tag{7}$$

where $|\psi'\rangle$ and $|\psi''\rangle$ are the normalized projections of $|\psi\rangle$ onto $H_0^{(1)}\otimes H^{(2)}$ and onto the complement $(H^{(1)}-H_0^{(1)})\otimes H^{(2)}$, respectively. Then the probabilities of local measurements performed on part 1 after the state verification measurement may depend on $|\psi\rangle$ only via its component $\beta|\psi''\rangle$. We will express this property in the following theorem.

Theorem 2. Let

$$|\psi\rangle=\alpha|\psi'\rangle+\beta|\psi''\rangle \ ,$$

where

$$|\psi'\rangle\in H_0^{(1)}\otimes H^{(2)} \ , \quad |\psi''\rangle\in(H^{(1)}-H_0^{(1)})\otimes H^{(2)} \ .$$

Then,

$$p(\psi)=|\alpha|^2p(\psi_0)+|\beta|^2p(\psi'') \ . \tag{8}$$

The erasing of local information about the part of the initial state in $H_0^{(1)}\otimes H^{(2)}$ is the essential property of the state verification measurement; the sensitivity of local measurements to $|\psi''\rangle$ and to $|\beta|$ is trivial. Indeed, the aim of the state verification measurement is to distinguish between $|\psi_0\rangle$ and the states orthogonal to it. But in order to distinguish between $|\psi_0\rangle$ and the states belonging to $(H^{(1)}-H_0^{(1)})\otimes H^{(2)}$ it is enough to perform a local measurement in part 1, and, similarly, a local measurement performed on part 2 can distinguish between $|\psi_0\rangle$ and the states belonging to $H^{(1)}\otimes(H^{(2)}-H_0^{(2)})$. Only for distinguishing between $|\psi_0\rangle$ and other states in the subspace $H_0^{(1)}\otimes H_0^{(2)}$ is a genuine nonlocal measurement needed. But this measurement can be performed by an interaction applying only to the $H_0^{(1)}\otimes H_0^{(2)}$ subspace, not the complementary subspaces $(H^{(1)}-H_0^{(1)})\otimes H^{(2)}$ and $H^{(1)}\otimes(H^{(2)}-H_0^{(2)})$.

The rest of this section is devoted to the proof of the above two theorems. We shall start with the proof of a simple property of state verification measurement. As follows from its definition, the measurement is *reliable,* that is, whenever the system is in $|\psi_0\rangle$, the answer is always "yes," while whenever the system is in an orthogonal state $|\psi_\perp\rangle$, the answer is always "no." Then,

$$\langle\phi|\langle\psi_\perp|U^\dagger\mathcal{P}_a^{(1)}U|\psi_0\rangle|\phi\rangle=0 \ , \tag{9}$$

where again U is the unitary transformation describing the state verification measurement, $\mathcal{P}_a^{(1)}$ is the projection operator on a certain outcome of the subsequent local measurement performed in part 1, and $|\phi\rangle$ is the initial state of the measuring device. Indeed, $U|\psi_0\rangle|\phi\rangle$ corresponds to "yes" states of the measuring device, while $U|\psi_\perp\rangle|\phi\rangle$ corresponds to "no" states. The operator $\mathcal{P}_a^{(1)}$ does not act on the states of the measuring device, therefore $\mathcal{P}_a^{(1)}U|\psi_0\rangle|\phi\rangle$ also belongs to a subspace of states corresponding to the answer "yes." Consequently, $\mathcal{P}_a^{(1)}U|\psi_0\rangle|\phi\rangle$ must be orthogonal to $U|\psi_\perp\rangle|\phi\rangle$.

We proceed now to the proof of Theorem 1 by dividing it into a lemma and two propositions.

Lemma. Let $|\psi\rangle=\alpha|\psi_0\rangle+\beta|\psi_\perp\rangle$, $\beta\neq 0$, where $|\psi_\perp\rangle$ is orthogonal to $|\psi_0\rangle$. Then, $p(\psi)=p(\psi_0)$ if and only if $p(\psi_\perp)=p(\psi_0)$.

Proof. Using Eq. (3) we obtain

$$\begin{aligned}p(\psi)&=\langle\phi|(\alpha^*\langle\psi_0|+\beta^*\langle\psi_\perp|)U^\dagger\mathcal{P}_a^{(1)}U(\alpha|\psi_0\rangle+\beta|\psi_\perp\rangle)|\phi\rangle\\&=|\alpha|^2p(\psi_0)+|\beta|^2p(\psi_\perp)+\alpha\beta^*\langle\phi|\langle\psi_\perp|U^\dagger\mathcal{P}_a^{(1)}U|\psi_0\rangle|\phi\rangle+\alpha^*\beta\langle\phi|\langle\psi_0|U^\dagger\mathcal{P}_a^{(1)}U|\psi_\perp\rangle|\phi\rangle \ .\end{aligned} \tag{10}$$

Now, from the reliability requirement (9) it follows that the last two terms vanish, and using the normalization condition $|\alpha|^2+|\beta|^2=1$, we finally obtain

$$p(\psi)=p(\psi_0)+|\beta|^2[p(\psi_\perp)-p(\psi_0)] . \tag{11}$$

Thus, $p(\psi)=p(\psi_0)$ if and only if $p(\psi_\perp)=p(\psi_0)$.

Using the lemma we now prove the following proposition.

Proposition 1. If the initial state of the system (prior to the state verification) can be expressed as a linear superposition,

$$|\psi\rangle=\sum_i^N c_i U_i^{(2)}|\psi_0\rangle , \tag{12}$$

where $U_i^{(2)}$ are unitary transformations in part 2 of the system, then the probabilities for the results of local measurements, performed in part 1 after the state verification, are equal to those obtained if the initial state were $|\psi_0\rangle$

$$p\left[\sum_i^N c_i U_i^{(2)}\psi_0\right]=p(\psi_0) . \tag{13}$$

Proof. We shall prove Eq. (13) by induction on N, the number of terms in the linear superposition. When $N=1$, Eq. (13) is true because it reduces to the causality condition (2). Let us now assume that Eq. (13) is true for $N=n$ and let us prove that it holds for $N=n+1$. Let $[U_{n+1}^{(2)}]^{-1}$ be the inverse of the unitary operator $U_{n+1}^{(2)}$. Then, from the causality principle (2), we obtain

$$\begin{aligned} p\left[\sum_i^{n+1} c_i U_i^{(2)}\psi_0\right] &= p\left[[U_{n+1}^{(2)}]^{-1}\sum_i^{n+1} c_i U_i^{(2)}\psi_0\right] \\ &= p\left[\sum_i^{n} c_i [U_{n+1}^{(2)}]^{-1}U_i^{(2)}\psi_0+c_{n+1}\psi_0\right] . \end{aligned} \tag{14}$$

Consider now the state $\mathcal{N}\sum_i^n c_i[U_{n+1}^{(2)}]^{-1}U_i^{(2)}\psi_0$. Here $\mathcal{N}$ is a normalization factor, appearing because $\sum_i^n c_i[U_{n+1}^{(2)}]^{-1}U_i^{(2)}\psi_0+c_{n+1}\psi_0$ is normalized. Since the $[U_{n+1}^{(2)}]^{-1}U_i^{(2)}$ are unitary transformations for all i, it follows from the induction assumption that

$$p\left[\mathcal{N}\sum_i^n c_i[U_{n+1}^{(2)}]^{-1}U_i^{(2)}\psi_0\right]=p(\psi_0) . \tag{15}$$

Let us now decompose

$$\mathcal{N}\sum_i^n c_i[U_{n+1}^{(2)}]^{-1}U_i^{(2)}|\psi_0\rangle=\alpha|\psi_0\rangle+\beta|\psi_\perp\rangle , \tag{16}$$

where $|\psi_\perp\rangle$ is orthogonal to $|\psi_0\rangle$. Then, from Eqs. (15) and (16) and the lemma, it follows that $p(\psi_\perp)=p(\psi_0)$.

Returning now to Eq. (14), we note that the state appearing in the last term can be decomposed as

$$\sum_i^n c_i[U_{n+1}^{(2)}]^{-1}U_i^{(2)}|\psi_0\rangle+c_{n+1}|\psi_0\rangle = \left[\frac{\alpha}{\mathcal{N}}+c_{n+1}\right]|\psi_0\rangle+\frac{\beta}{\mathcal{N}}|\psi_\perp\rangle .$$

Since we have already established that $p(\psi_\perp)=p(\psi_0)$, using the lemma again we obtain

$$p\left[\sum_i^n c_i[U_{n+1}^{(2)}]^{-1}U_i^{(2)}\psi_0+c_{n+1}\psi_0\right]=p(\psi_0) . \tag{17}$$

Inserting (17) into (14) ends the proof of Eq. (13) and of the proposition.

To complete the proof of Theorem 1, we have to prove the second proposition.

Proposition 2. Any state $|\psi\rangle$ which belongs to the Hilbert space $H_0^{(1)}\otimes H^{(2)}$ can be expressed in the form of Eq. (12), i.e., $|\psi\rangle=\sum_i^N c_i U_i^{(2)}|\psi_0\rangle$.

Proof. Let $\{|i\rangle_1|j\rangle_2\}$ be the basis of the Schmidt decomposition (6). To prove the proposition it is enough to show that by superpositions of the form (12) we can obtain any vector $|p\rangle_1|q\rangle_2$ of this basis belonging to the subspace $H_0^{(1)}\otimes H^{(2)}$.

Consider the unitary transformations $V_1^{(2)}$ and $V_2^{(2)}$ defined by

$$\begin{aligned} &V_1^{(2)}|p\rangle_2=|q\rangle_2 , \\ &V_1^{(2)}|q\rangle_2=|p\rangle_2 , \\ &V_1^{(2)}|k\rangle_2=|k\rangle_2 \quad \text{for } k\neq p,q , \end{aligned} \tag{18a}$$

$$\begin{aligned} &V_2^{(2)}|p\rangle_2=-|q\rangle_2 , \\ &V_2^{(2)}|q\rangle_2=|p\rangle_2 \quad \text{if } q\neq p , \\ &V_2^{(2)}|k\rangle_2=|k\rangle_2 \quad \text{for } k\neq p,q . \end{aligned} \tag{18b}$$

Then,

$$\frac{1}{2\alpha_p}V_1^{(2)}|\psi_0\rangle-\frac{1}{2\alpha_p}V_2^{(2)}|\psi_0\rangle=|p\rangle_1|q\rangle_2 , \tag{19}$$

where α_p is the corresponding coefficient in the Schmidt decomposition (6). This ends the proof of the proposition.

The proof of the above two propositions completes the proof of the theorem. Indeed, Proposition 1 says that for any state $|\psi\rangle$ which can be expressed in the form of Eq. (12) we have $p(\psi)=p(\psi_0)$, and Proposition 2 says that any state which belongs to the subspace $H_0^{(1)}\otimes H^{(2)}$ can be expressed in the form of Eq. (12). Thus, if $|\psi\rangle\in H_0^{(1)}\otimes H^{(2)}$, then $p(\psi)=p(\psi_0)$.

Using Theorem 1 we will now prove Theorem 2.

Proof. Using Eq. (3), we obtain

$$\begin{aligned} p(\psi)&=\langle\phi|(\alpha^*\langle\psi'|+\beta^*\langle\psi''|)U^\dagger\mathcal{P}_a^{(1)}U(\alpha|\psi'\rangle+\beta|\psi''\rangle)|\phi\rangle \\ &=|\alpha|^2p(\psi')+|\beta|^2p(\psi'')+\alpha\beta^*\langle\phi|\langle\psi''|U^\dagger\mathcal{P}_a U|\psi'\rangle|\phi\rangle+\alpha^*\beta\langle\phi|\langle\psi'|U^\dagger\mathcal{P}_a^{(1)}U|\psi''\rangle|\phi\rangle . \end{aligned} \tag{20}$$

Theorem 1 implies that $p(\psi')=p(\psi_0)$; therefore, we have only to show that the last two terms of Eq. (20) vanish. Since these terms are complex conjugates of one another, it is enough to prove that, say, the first of the two vanishes. Let us calculate this term using Proposition 2, i.e., the fact that since $|\psi'\rangle \in H_0^{(1)} \otimes H^{(2)}$ it has the form of Eq. (12)

$$\alpha\beta^* \langle \phi | \langle \psi'' | U^\dagger \mathcal{P}_a U | \psi' \rangle | \phi \rangle = \alpha\beta^* \sum_i^N c_i \langle \phi | \langle \psi'' | U^\dagger \mathcal{P}_a U U_i^{(2)} | \psi_0 \rangle | \phi \rangle \ . \quad (21)$$

Now we shall show that each term in the last sum is equal to zero. Using the causality principle as in Eq. (5) and taking the unitary transformation acting on part 2 to be $[U_i^{(2)}]^{-1}$, we obtain

$$\langle \phi | \langle \psi'' | U^\dagger \mathcal{P}_a U U_i^{(2)} | \psi_0 \rangle | \phi \rangle = \langle \phi | \langle \psi'' | U_i^{(2)} U^\dagger \mathcal{P}_a^{(1)} U | \psi_0 \rangle | \phi \rangle \ . \quad (22)$$

Since $|\psi''\rangle \in (H^{(1)} - H_0^{(1)}) \otimes H^{(2)}$ and since $[U_i^{(2)}]^{-1}$ acts only in part 2, we have also $[U_i^{(2)}]^{-1} |\psi''\rangle \in (H^{(1)} - H_0^{(1)}) \otimes H^{(2)}$, and therefore the state $[U_i^{(2)}]^{-1} |\psi''\rangle$ is orthogonal to $|\psi_0\rangle$. Now, Eq. (9) implies that the right-hand side of Eq. (22) vanishes, i.e., each individual term in Eq. (21) vanishes. This ends the proof of Theorem 2.

V. OPERATOR MEASUREMENTS

We shall now use the result of the previous section in the study of standard quantum measurements. The measurement of an operator A can be considered a verification measurement of each of its nondegenerate eigenstates. It immediately follows that most operators having some nondegenerate eigenstates are unmeasurable. Indeed, on the one hand, the final state of the system must be locally independent of its initial state, as follows from Theorems 1 and 2. On the other hand, if the system is initially in an eigenstate, it should be undisturbed by the measurement. Only in very special cases can these two requirements be simultaneously satisfied.

Let us consider the simplest nonlocal system, two nonidentical spin-$\frac{1}{2}$ particles separated in space. Let $|\psi_0\rangle$ be an arbitrary entangled state of these particles. We shall prove that the projection operator onto $|\psi_0\rangle$, $\mathcal{P}_{|\psi_0\rangle}$, is unmeasurable. Choosing appropriate local bases, we can write $|\psi_0\rangle$ (Schmidt decomposition) as

$$|\psi_0\rangle = \alpha|\uparrow_z\rangle|\uparrow_{z'}\rangle + \beta|\downarrow_z\rangle|\downarrow_{z'}\rangle \ , \quad (23)$$

where $\alpha, \beta \neq 0$, and the arrows represent the spin polarized "up" or "down" along some arbitrary directions z and z'. Consider now two possible initial states $|\psi_1\rangle = |\uparrow_z\rangle|\downarrow_{z'}\rangle$ and $|\psi_2\rangle = |\downarrow_z\rangle|\uparrow_{z'}\rangle$, and let us suppose that $\mathcal{P}_{|\psi_0\rangle}$ is measurable. Then, as $|\psi_1\rangle$ and $|\psi_2\rangle$ are both eigenstates of $\mathcal{P}_{|\psi_0\rangle}$ (corresponding to the eigenvalue zero), they must not be disturbed by the measurement, so the system will end in $|\psi_1\rangle$ or $|\psi_2\rangle$, respectively, which are locally distinguishable. But the measurement of $\mathcal{P}_{|\psi_0\rangle}$ is a verification of $|\psi_0\rangle$, and according to Theorem 1, which applies in this case, it must erase all local information. The projection operator $\mathcal{P}_{|\psi_0\rangle}$ is thus unmeasurable.

We shall now analyze the measurability of completely nondegenerate spin operators. We state our result in the following theorem.

Theorem 3. Causality constrains measurements of nondegenerate spin operators of a composite system on two spin-$\frac{1}{2}$ particles such that the only measurable operators are those with eigenstates of two possible types,

$$\begin{aligned} |\psi_1\rangle &= |\uparrow_z\rangle_1 |\uparrow_{z'}\rangle_2 \ , \\ |\psi_2\rangle &= |\uparrow_z\rangle_1 |\downarrow_{z'}\rangle_2 \ , \\ |\psi_3\rangle &= |\downarrow_z\rangle_1 |\uparrow_{z'}\rangle_2 \ , \\ |\psi_4\rangle &= |\downarrow_z\rangle_1 |\downarrow_{z'}\rangle_2 \ , \end{aligned} \quad (24a)$$

or

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1 |\uparrow_{z'}\rangle_2 + |\downarrow_z\rangle_1 |\downarrow_{z'}\rangle_2) \ , \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1 |\uparrow_{z'}\rangle_2 - |\downarrow_z\rangle_1 |\downarrow_{z'}\rangle_2) \ , \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1 |\downarrow_{z'}\rangle_2 + |\downarrow_z\rangle_1 |\uparrow_{z'}\rangle_2) \ , \\ |\psi_4\rangle &= \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1 |\downarrow_{z'}\rangle_2 - |\downarrow_z\rangle_1 |\uparrow_{z'}\rangle_2) \ . \end{aligned} \quad (24b)$$

The actual eigenvalues are irrelevant; they must only be different from each other, so that the operator is completely nondegenerate.

Proof. Operators of type (24a), although referring to both spins, are effectively local. They can be measured by simply measuring the z component of the spin of the first particle and the z' component of the spin of the second particle. Operators of the type (24b) are truly nonlocal, since they have entangled eigenstates. In fact, the eigenstates (24b) are all maximally entangled. The measurability of these operators has been shown [2,3] and an explicit measuring method, involving only local interactions, has been given. They provided, in fact, the first example of nonlocal variables which can be instantaneously measured in the framework of relativistic quantum mechanics.

What remains to be proven is that if the eigenstates of a nondegenerate operator cannot be brought to either of the forms (24a) or (24b), then causality forbids its measurement. Consider first a nondegenerate operator A, for which all its eigenstates are direct products. Up to an interchange of the roles of particles 1 and 2, the set of eigenstates of such an operator can always be written as

$$\begin{aligned} |\psi_1\rangle &= |\uparrow_z\rangle_1 |\uparrow_{z'}\rangle_2 \ , \\ |\psi_2\rangle &= |\downarrow_z\rangle_1 |\uparrow_{z'}\rangle_2 \ , \\ |\psi_3\rangle &= |\uparrow_{z''}\rangle_1 |\downarrow_{z'}\rangle_2 \ , \\ |\psi_4\rangle &= |\downarrow_{z''}\rangle_1 |\downarrow_{z'}\rangle_2 \ . \end{aligned} \quad (25)$$

If z'' is parallel or antiparallel to z then the set of eigenstates (25) is equivalent to the set (24a). Let us prove from causality that, indeed, z'' must be parallel or antiparallel to z.

Let us write $p(\psi)$ for probability to obtain $\sigma_z^{(1)}=-1$ in a measurement performed on particle 1 immediately after a measurement of A when $|\psi\rangle$ is the initial state of the system. Consider two possible initial states of the system,

$$|\xi_1\rangle=|\psi_1\rangle=|\uparrow_z\rangle_1|\uparrow_{z'}\rangle_2\ ,\qquad |\xi_2\rangle=|\uparrow_z\rangle_1|\downarrow_{z'}\rangle_2\ . \tag{26}$$

From the causality principle it follows that

$$p(\xi_1)=p(\xi_2)\ . \tag{27}$$

The state $|\xi_1\rangle$ is an eigenstate of A. Thus, the measurement of A does not disturb this state, and, therefore, the probability to obtain $\sigma_z^{(1)}=-1$ afterwards vanishes, $p(\xi_1)=0$.

On the other hand,

$$p(\xi_2)=\sum_i|\langle\xi_2|\psi_i\rangle|^2p(\psi_i)=|\langle\uparrow_z|\uparrow_{z''}\rangle|^2|\langle\downarrow_z|\uparrow_{z''}\rangle|^2+|\langle\uparrow_z|\downarrow_{z''}\rangle|^2|\langle\downarrow_z|\downarrow_{z''}\rangle|^2\ . \tag{28}$$

From the right-hand side of (28) we see that, indeed, $p(\xi_2)=0$ if, and only if, z'' is parallel or antiparallel to z. This ends the proof that if the eigenstates of a measurable nondegenerate operator are direct products, they also have the form (24a).

Consider now an operator A, which has at least one entangled nondegenerate eigenstate, say $|\psi_1\rangle$. By choosing appropriate local bases we can write

$$|\psi_1\rangle=\alpha|\uparrow_z\rangle_1|\uparrow_{z'}\rangle_2+\beta|\downarrow_z\rangle_1|\downarrow_{z'}\rangle_2\ . \tag{29}$$

We now regard the measurement of A as a verification of the state $|\psi_1\rangle$. For an entangled state, both α and β are nonzero and Theorem 1 implies that the measurement of A erases from the system all local information about its initial state. Since the eigenstates of A are undisturbed by the measurement, they must be locally indistinguishable. This requirement can be fulfilled only if $|\alpha|=|\beta|=1/\sqrt{2}$. Indeed, if $|\alpha|\neq|\beta|$ there are no states orthogonal to $|\psi_1\rangle$ and locally indistinguishable from it. On the other hand, when $|\alpha|=|\beta|=1/\sqrt{2}$, the requirement of local indistinguishability implies that the eigenstates have the form

$$|\psi_i\rangle=\frac{1}{\sqrt{2}}(|\uparrow_{z_i}\rangle_1|\uparrow_{z_i'}\rangle_2+e^{i\phi_i}|\downarrow_{z_i}\rangle_1|\downarrow_{z_i'}\rangle_2)\ , \tag{30}$$

up to overall irrelevant phases. Note that the directions z and z' depend on i, and they must be chosen so that the states $|\psi_i\rangle$ are mutually orthogonal.

It remains to be proven that any four mutually orthogonal states (30) can be brought, by choosing appropriate local bases, to the form (24b). For simplicity, let us first redefine the base vectors such that $|\psi_1\rangle$ reads

$$|\psi_1\rangle=\frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1|\downarrow_{z'}\rangle_2-|\downarrow_z\rangle_1|\uparrow_{z'}\rangle_2)\ . \tag{31}$$

The above form of $|\psi_1\rangle$ has the property that it is invariant when we change the basis vectors in an identical way for both particles. Consider now the eigenstate $|\psi_2\rangle$, which is orthogonal to $|\psi_1\rangle$ and locally indistinguishable from it. Expressed in the same local basis as (31), the most general form of such a state (up to an overall phase) is

$$|\psi_2\rangle=\frac{1}{\sqrt{2}}\cos\alpha(e^{i(\phi/2)}|\uparrow_z\rangle_1|\uparrow_{z'}\rangle_2+e^{-i(\phi/2)}|\downarrow_z\rangle_1|\downarrow_{z'}\rangle_2)+\frac{i}{\sqrt{2}}\sin\alpha(|\uparrow_z\rangle_1|\downarrow_{z'}\rangle_2+|\downarrow_z\rangle_1|\uparrow_{z'}\rangle_2)\ . \tag{32}$$

Consider now the local basis transformations given implicitly by

$$|\uparrow_z\rangle=e^{-i(\phi/2)}(\cos\beta|\uparrow_\xi\rangle+i\sin\beta|\downarrow_\xi\rangle)\ ,\qquad |\downarrow_z\rangle=e^{i(\phi/2)}(i\sin\beta|\downarrow_\xi\rangle+\cos\beta|\uparrow_\xi\rangle)\ , \tag{33a}$$

$$|\uparrow_{z'}\rangle=e^{-i(\phi/2)}(\cos\beta|\uparrow_{\xi'}\rangle+i\sin\beta|\downarrow_{\xi'}\rangle)\ ,\qquad |\downarrow_{z'}\rangle=e^{i(\phi/2)}(i\sin\beta|\downarrow_{\xi'}\rangle+\cos\beta|\uparrow_{\xi'}\rangle)\ , \tag{33b}$$

where $\tan2\beta=\cot\alpha$. These transformations preserve the form of $|\psi_1\rangle$,

$$|\psi_1\rangle=\frac{1}{\sqrt{2}}(|\uparrow_\xi\rangle_1|\downarrow_{\xi'}\rangle_2-|\downarrow_\xi\rangle_1|\uparrow_{\xi'}\rangle_2)\ , \tag{34}$$

and bring $|\psi_2\rangle$ to the form

$$|\psi_2\rangle=\frac{1}{\sqrt{2}}(|\uparrow_\xi\rangle_1|\downarrow_{\xi'}\rangle_2+|\downarrow_\xi\rangle_1|\uparrow_{\xi'}\rangle_2)\ . \tag{35}$$

In this new base, the most general form of $|\psi_3\rangle$ (orthogonal to and locally indistinguishable from $|\psi_1\rangle$ and $|\psi_2\rangle$) is

$$|\psi_3\rangle=\frac{1}{\sqrt{2}}(|\uparrow_\xi\rangle_1|\uparrow_{\xi'}\rangle_2+e^{i\sigma}|\downarrow_\xi\rangle_1|\downarrow_{\xi'}\rangle_2)\ . \tag{36}$$

By redefining the phases of the base vectors,

$$|\uparrow_\xi\rangle\to e^{i(\sigma/2)}|\uparrow_\xi\rangle\ ,\qquad |\downarrow_\xi\rangle\to e^{-i(\sigma/2)}|\downarrow_\xi\rangle\ , \tag{37a}$$

$$|\uparrow_{\xi'}\rangle\to e^{i(\sigma/2)}|\uparrow_{\xi'}\rangle\ ,\qquad |\downarrow_{\xi'}\rangle\to e^{-i(\sigma/2)}|\downarrow_{\xi'}\rangle\ , \tag{37b}$$

we preserve the form of $|\psi_1\rangle$ and $|\psi_2\rangle$ and eliminate the relative phase between the two terms in (36), and thus obtain (up to an overall phase)

$$|\psi_3\rangle=\frac{1}{\sqrt{2}}(|\uparrow_\xi\rangle_1|\uparrow_{\xi'}\rangle_2+|\downarrow_\xi\rangle_1|\downarrow_{\xi'}\rangle_2)\ . \tag{38}$$

Finally, the eigenstate $|\psi_4\rangle$ is determined by its orthogonality to $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$, and takes the form

$$|\psi_4\rangle=\frac{1}{\sqrt{2}}(|\uparrow_\xi\rangle_1|\uparrow_{\xi'}\rangle_2-|\downarrow_\xi\rangle_1|\downarrow_{\xi'}\rangle_2)\ . \tag{39}$$

This completes our proof, since the set of eigenstates $|\psi_1\rangle,\ldots,|\psi_4\rangle$ is, up to renumbering, equivalent to the set (24b).

VI. NONDEMOLITION VERIFICATIONS AND IDEAL MEASUREMENTS OF THE FIRST KIND

As a final application of Theorem 1, we will study the possibility of performing *ideal measurements of the first kind.* A basic assumption in axiomatic nonrelativistic quantum theory [4] is that every property of a quantum system may be determined via an ideal measurement of the first kind. We will now show that there are nonlocal properties which cannot be determined in this way.

Ideal measurements of the first kind are a particular case of *nondemolition* measurements. A *nondemolition verification* of the state $|\psi_0\rangle$ is a state verification measurement with an additional requirement that, if the result of the measurement is "yes," then the system ends up in the state $|\psi_0\rangle$. In particular, if the system is initially in the state $|\psi_0\rangle$, it will remain in this state. On the other hand if the result is "no," then there are no restrictions on what will be the final state of the system. Analogously, we define *verifications of higher-dimensional Hilbert subspaces;* a verification of a subspace R is a measurement which always yields "yes" if the state of the system belongs to R and "no" if the state is orthogonal to R. As in the case of state verification, no restrictions are imposed on the state of the system after the measurement. A *nondemolition* verification of R has the supplementary property that if the result of the measurement is "yes," the final state is the projection of the initial state on R. In particular, if the initial state belongs to R, the state remains unchanged.

It has already been shown that any state can be verified in a nondemolition way (exchange measurements [3]). Thus, it is possible to perform ideal measurement of the first kind to verify an arbitrary state. However, this is not the case for verification of higher-dimensional Hilbert spaces. We will now present an example of a three-dimensional Hilbert space which cannot be verified in a nondemolition way, and, therefore, no corresponding ideal measurement of the first kind is possible. Consider once again two nonidentical spin-$\frac{1}{2}$ particles. Let R be the subspace of states which are orthogonal to the state

$$|\psi_0\rangle=\alpha|\uparrow\rangle|\uparrow\rangle+\beta|\downarrow\rangle|\downarrow\rangle\ , \quad \alpha,\beta\neq 0\ . \tag{40}$$

The proof that R cannot be verified in a nondemolition way is identical to the proof of the unmeasurability of the projector on $|\psi_0\rangle$. Since $|\psi_0\rangle$ is the unique state orthogonal to R, a nondemolition verification of R is at the same time a verification (not necessarily nondemolition) of $|\psi_0\rangle$. Thus we can apply Theorem 1, i.e., all local information must be erased. Consider, however, two possible initial states,

$$|\psi_1\rangle=|\uparrow\rangle|\downarrow\rangle\ , \quad |\psi_2\rangle=|\downarrow\rangle|\uparrow\rangle\ . \tag{41}$$

Both $|\psi_1\rangle$ and $|\psi_2\rangle$ belong to R and, therefore, should be unaffected by the measurement. But since they are locally distinguishable, they will lead to locally distinguishable final states, in contradiction to Theorem 1. This ends our proof.

VII. CONCLUSIONS

We have proved that even according to the weakest definition of state verification, requiring only reliability of the measurement, causality implies that verification of an entangled state must erase local information. We have analyzed conditions for which all local information must be erased by the state verification and have found that there is a very wide class of such situations (see Theorem 1). An example is a verification measurement of *any* entangled state of two spin-$\frac{1}{2}$ particles. The causality principle states that any disturbance of a particle just prior to a time t_0 cannot affect the results of local measurements performed on a second particle immediately after t_0. We, however, have proved the surprising result that also any disturbance of the *second* particle before t_0 does not change probabilities for the results of local measurements performed on that particle after verification of an entangled state at t_0.

We have also shown in general what local information must be erased by verification of an entangled state (Theorem 2). These theorems helped us analyze the question of measurability of operators. We completely analyzed the measurability of nondegenerate spin operators on a system of two spin-$\frac{1}{2}$ particles (Theorem 3). We have shown that causality imposes severe constraints. Even certain local operators, i.e., operators with product eigenstates [but not of the type (24a)], cannot be measured without violating causality. However, there are operators with entangled eigenstates that can be measured (25b). Measurability of all but two types of operators contradicts the causality principle. For the operators of these two types, there are known measurement procedures that use only local (and, therefore, causal) interactions.

We applied Theorem 1 to show that for certain Hilbert subspaces, there is no way to perform an ideal measurement of the first kind without violating causality. This raises new difficulties for the construction of a relativistic axiomatic quantum theory as an extension of the nonrelativistic one.

We hope that our investigation of the constraints on measuring nonlocal variables due to relativistic causality can be extended to more general situations, and that it will lead to a better understanding of the relativistic quantum theory of measurement.

Note added. Recently we learned that Bennett *et al.* [8] have found a method for teleportation of quantum states. A similar method can serve as an alternative to nonlocal exchange measurements. In this method the local information is also completely erased in accordance with our Theorem 1.

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